Self-Triggered Set-Valued Observers

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Abstract—This paper addresses the problem of high computational requirements in the implementation of Set-Valued Observers (SVOs), which places stringent constraints in terms of their use in applications where low computational power is available or the plant is sensitive to delay. It is firstly shown how to determine an overbound for the set-valued estimates, which reduces the overhead by limiting the number of inequalities defining those set-valued state estimates. In the particular setting of distributed gossip problems, the proposed algorithm is shown to have constant complexity. This algorithm is of prime importance to reduce the computational load and enable the use of such estimates for real-time applications. Results are also provided regarding the frequency of the triggers in the worst-case scenario. The performance of the proposed method is evaluated through simulation.

I. INTRODUCTION

The problem of fault detection for general linear systems relates to that of determining, for any given set of inputs, noise and disturbances, if the measurements of the system can be produced by the model. An interesting instance of the problem is detecting faults in an asynchronous distributed environment, which refers to determining if any node enters an incoherent state given the observed measurements.

A special case of interest rises in the domain of randomized distributed algorithms, both due to their relevance in certain problems but also because of their unstructured nature, i.e., all nodes play the same role in the algorithm and the messages need not satisfy any particular type of time sequence. This class of algorithms is used for iterative solutions, because they are robust against packet drops and node failure. Applications of randomized algorithms range from selection and sorting [1] to consensus [2] and problems for which the solution requires a heavy computational burden. However, a set-valued estimate of the state is computed by union of the set of possible states generated by each transmission. By definition, the number of sets grows exponentially with the number of past time instants, i.e., the horizon N .

The study of Fault Detection and Isolation (FDI) problems has been a long standing research topic, since the early 70's

(see [3]), but still poses remarkable challenges to both the scientific community and the industry (see, for example, the survey in [4] and references therein). Classical fault detection methods such as the ones proposed in [3], [5] and [6], rely on designing filters that generate residuals that should be *large* under faulty environments. Calculating thresholds for the residuals is typically cumbersome or poses stringent assumptions on the exogenous disturbances and measurement noise acting upon the system.

An alternative approach, based on Set-Valued Observers (SVOs) was described in [7]. The concept of SVOs was first introduced in [8] and [9], and further information can be found in [10], [11] and the references therein. The SVO-based solution alleviates the design complexity, while posing mild assumptions on the system. However, it also requires increased computational power when compared to classical FDI methodologies.

In the literature, another option is to use the concept of zonotopes, described in [12] and further developed in [13]. Those represent a different trade-off between intersections and unions of sets. One could also use the idea of interval analysis [14], although it introduces conservatism by not considering higher horizon values in their formulation, unlike the SVOs [7]. An alternative approach is adopted in this article, as described in the sequel.

The main contributions of this paper are as follows:

- given a specific structure for the matrix defining the polytope (i.e., the set-valued state estimate), it is shown how to compute an overbounding ellipsoid or ball;
- based on the concept of singular vectors, we show how a rotation can be found to prevent the approximation error of using boxes from going to infinity when the matrix defining the polytope is ill-conditioned;
- an algorithm is introduced that uses approximations to the *optimal* SVO estimates based on the previous methods, which is less computationally demanding, and self-triggers the computation of the aforementioned estimates only when necessary to ensure convergence;
- results are provided for the worst-case triggering frequency for a Linear Parameter-Varying (LPV) system;
- finally, for the special case of a distributed linear algorithm with a gossip property, it is shown that the overbounds are efficient to compute and propagate, since its complexity is constant.

The remainder of this paper is organized as follows. In Section II, we describe the SVO-based fault detection problem. The proposed solution to reduce the computational cost and limit the conservativeness is given in Section III. Section IV presents the algorithm for Self-Triggered SVOs, while the main properties of the results obtained are stated in Section V and illustrated in simulation in Section VI. Concluding remarks are provided in Section VII.

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Notation : The transpose of a matrix A is denoted by $A^\top$. We let $\mathbf{1}_n := [1 \dots 1]^\top$ and $\mathbf{0}_n := [0 \dots 0]^\top$ and I_n denotes the identity matrix of dimension n . Dimensions are omitted when clear from context. The vector $\mathbf{e}_i$ denotes the canonical vector whose components are equal to zero, except for the i th component. The notation $\|\cdot\|$ refers to $\|v\| := \sup_i |v_i|$ for a vector, and $\|A\| := \bar{\sigma}(A)$ and $\|\cdot\|_2 := \sqrt{v^\top v}$. The Orthogonal group of dimension n is represented by $O(n)$.

II. PROBLEM STATEMENT

We consider the dynamics of a Linear Parameter-Varying (LPV) system S of the form:

$$S : \begin{cases} x(k+1) = A(\Delta(k))x(k) + B(\Delta(k))u(k) \\ y(k) = C(\Delta(k))x(k) \end{cases} \quad (1)$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^{n_u}$, matrices $A(\Delta(k))$ represent the dynamics of the system and are taken from $\{Q_i, i = 1, \dots, m\}$ with each Q_i being the possible dynamics in each time instant. The signal u is designed so as to ensure the state remains bounded. We make the assumption that $A(\Delta(k)) = A_0 + A_\Delta(k)$ where

$$A_\Delta(k) = \sum_{\ell=1}^{n_\Delta} \Delta_\ell(k) A_\ell$$

where n_Δ is the number of required uncertainties and each $\Delta_\ell(k)$ is a scalar uncertainty with $|\Delta_\ell(k)| \leq 1$. The collection of matrices A_ℓ is sufficiently rich as to represent all matrices Q_i . We assume for simplicity that both matrices $B(\Delta(k))$ and $C(\Delta(k))$ are parameter-dependent but with no uncertainty, which can be relaxed by employing the techniques described in [15]. Whenever clear from context that matrices depend on the parameter $\Delta(k)$ we will drop the Δ and refer to the matrices as $A(k)$, $B(k)$ and $C(k)$.

We use the SVO framework from [15] and [16] and define, at transmission time k ,

$$X(k) := \text{Set}(M(k), m(k))$$

as the smallest set containing all possible state realizations at time instant k , where $\text{Set}(M, m) := \{q : Mq + m \leq 0\}$ represents a convex polytope, where the operator $\leq$ is understood component-wise. The aim of an SVO is to find an approximation of the smallest set containing all possible states of the system at time k , $\tilde{X}(k)$, with the knowledge that $\forall_{0 \leq i < N}$, $x(k-i) \in \tilde{X}(k-i)$ and that the dynamics of the system are as in (1).

More precisely, the initial state is assumed to satisfy $x(0) \in X(0)$ where $X(0) := \text{Set}(M_0, m_0)$ and we can select M_0 and m_0 such that the corresponding polytope is guaranteed to contain the initial state. If $\Delta(k) = \Delta^*$ is known, then the set $X(k+1) := \text{Set}(M(k+1), m(k+1))$, which contains all the possible states of the system at time $k+1$, can be described by the set of points, $\mathbf{x}$, satisfying

$$\underbrace{\begin{bmatrix} M(k)(A_0 + A_{\Delta^*})^{-1} \\ \bar{C}(k+1) \end{bmatrix}}_{M(k+1)} \mathbf{x} \leq \underbrace{\begin{bmatrix} -m(k) + \tilde{u}(k) \\ \bar{y}(k+1) \\ -m(k+1) \end{bmatrix}}_{-m(k+1)}. \quad (2)$$

where $\tilde{u}(k) := M(k)(A_0 + A_{\Delta^*})^{-1}B(k)u(k)$ and $\bar{Z} := \begin{bmatrix} Z \\ -Z \end{bmatrix}$. This procedure assumes an invertible transmission

matrix. When this is not the case, we can adopt the strategy in [17] and solve the inequality

$$\begin{bmatrix} \bar{I} & -\bar{A}_0 - \bar{A}_{\Delta^*} \\ \bar{C}(k+1) & 0 \\ 0 & M(k) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}^- \end{bmatrix} \leq \begin{bmatrix} \bar{B}(k)u(k) \\ \bar{y}(k+1) \\ -m(k) \end{bmatrix} \quad (3)$$

by applying the Fourier-Motzkin elimination method [18] to remove the dependence on $\mathbf{x}^-$ and obtain the set described by $M(k+1)\mathbf{x} \leq -m(k+1)$.

These calculations can be extended to the case where $\tilde{X}(k)$ is computed not only based on $\tilde{X}(k-1)$, but also based on previous estimates, $\tilde{X}(k-2), \dots, \tilde{X}(k-N)$, where N is the so-called horizon of the SVO. This method is based on standard *lifting* techniques and allows reducing the conservatism of the approach when $\tilde{X}(k-1)$ is larger than $X(k-1)$. For further details on this topic, the reader is referred to [15].

Let the coordinates of each vertex of the hypercube $H := \{\delta \in \mathbb{R}^{n_\Delta} : |\delta| \leq 1\}$ be denoted by $\theta_i, i = 1, \dots, 2^{n_\Delta}$. Using (3), we compute $X_{\theta_i}(k)$ with $\Delta^* = \theta_i$. Thus, the set of all possible states at time $k+1$ can be obtained by

$$\tilde{X}(k+1) = \text{co}\left(\bigcup_{\theta_i \in H} \text{Set}(M_{\theta_i}(k+1), m_{\theta_i}(k+1))\right) \quad (4)$$

where we make the union for all the values θ_i and where M_{θ_i} and m_{θ_i} are obtained using (3). The convex hull function, $\text{co}()$ of set $X(k+1)$ is then obtained by using the methods described in [19], since, in general, the set $X(k+1)$ is non-convex even if $X(k)$ is convex. For additional properties of the set $\tilde{X}(k)$, the interested reader is referred to [20] and the references therein.

An important issue regarding the SVOs is its computational burden, since the complexity grows exponentially both on the number of uncertainties n_Δ and also on the horizon N , since in (4), H is of size 2^{n_Δ} . This problem can render the SVOs inapplicable for some systems, especially those with stringent time constraints such as control in real-time applications.

The main goal of this paper can therefore be stated as: reduce the necessary number of calculations by only performing the SVO computations when necessary, according to some user-defined criterion, while lowering the conservatism of the solution.

III. ELLIPSOID OVERBOUNDING

The main issue when applying SVO-based techniques to a real-time or time-sensitive application is its computational burden. In each iteration, computing the union of the sets obtained by propagating all possible combinations of the system dynamics and intersecting it with the set of states compatible with the current measurements is time-consuming due to the number of possible combinations - see [20].

In this section, we assume the particular structure $\tilde{X}(k) = \{q : M(k)q \leq 1\}$, where matrix $M(k)$ takes the form

$$M(k) = [\bar{M}_1(k)^\top \quad \dots \quad \bar{M}_\ell(k)^\top]^\top \quad (5)$$

and where the polytope $\tilde{X}(k)$ is defined as the intersection of 2ℓ half-planes.

Theorem 1: Consider a convex set

$$S = \{x : Mx \leq 1\}$$

such that M can be written as in (5). An ellipsoidal overbound to $\mathcal{S}$ is given by

$$x^\top P x \leq 1,$$

where $P = \frac{VS^\top SV^\top}{n}$.

Proof: We have that $Mx \leq 1_n$ and, since matrix M can be written in the format (5), we have the implication $\forall x : Mx \leq 1_n \Rightarrow x^\top M^\top M x \leq 1_n^\top 1_n$. Performing a singular-valued decomposition on M we get $\frac{1}{n} x^\top VS^\top U^\top U S V^\top \leq 1$ and, since U is a unitary matrix, we get the conclusion. ■

A corollary of the previous result can be derived in order to provide an overbound in terms of the maximum norm of any point belonging to the set.

Corollary 2: Consider a convex set $\mathcal{S}$ such that matrix M can be written as in (5). Then, we can overbound $\mathcal{S}$ by a ball described by

$$x^\top x \leq \frac{n}{\sigma_{\min}^2(M)}$$

which means that $\|x\|_2 \leq \frac{\sqrt{(n)}}{\sigma_{\min}(M)}$.

Proof: Consider the previous result where we have upperbounded the set $\mathcal{S}$ by an ellipse. By upperbounding the singular values, we get $P = \frac{\sigma_{\min}^2}{n} V I V^\top = \frac{\sigma_{\min}^2}{n} I$ which leads to the result. ■

Remark 3: It is stressed that alternative methods have been developed in the literature to obtain ellipsoidal set-valued state estimates, since the seminal work by [9], as described, for instance, in [10]. However, the algorithm proposed in this work has some relevant properties, as discussed in the sequel, including the low-computational power required, as well as guaranteeing that the state of the system is indeed contained within the ellipsoid.

IV. SELF-TRIGGERED SET-VALUED OBSERVERS

Self-Triggered SVOs aim to reduce the computational cost associated with the *classical* SVOs by only computing the set-valued state estimate (using the previously described tools) when the set grows in size above a certain threshold.

The main advantage of this method is its real-time application due to diminished computational costs associated with three main factors:

- The matrix defining the polytope generally belongs to $\mathbb{R}^{\ell \times n}$, where $\ell \gg n$ and represents the number of restrictions associated with the edges of the polytope whereas the proposed overbound matrix belongs to $\mathbb{R}^{n \times n}$;
- Running the SVO computations at some time instants allows to use the idle moments to pre-compute the necessary combinations of matrices products;
- In some special cases of interest, such as in distributed systems, it is possible to discard dynamics matrices based on the observation set and compute the worst estimate with minimal processing effort.

Start by noticing that the first rows of (2) are

$$M(k)(A_0 + A_{\Delta^*})^{-1} \mathbf{x} \leq -m(k) \quad (6)$$

where (6) defines the set of points that originate from propagating the previous points of the set defined by $\text{Set}(M(k), m(k))$. If the sets are always defined to have

Fig. 1: Original set and ellipsoidal overbound with the set resulting from the intersection with the measurement set to form the new set-valued estimate.

$m(k) = -1_n$ then we are in the conditions to bound the state at time k with Theorem 1 and propagate it using the dynamics of the system, considering each given instantiation of the uncertainties in Δ^* . The assumption only forces the set to include the origin, although it is not very stringent as we can always perform a translation. Thus, $X(k+1) = \{(A_0 + A_{\Delta^*})^{-1} x \wedge x \in X(k)\}$.

From the previous discussion, the set $X(k+1)$ can be described as

$$X(k+1) = \bigcup_{\Delta \in H} \text{Set}(M(k)(A_0 + A_{\Delta}(k))^{-1}, -1) \quad (7)$$

If the original set defined by the matrix $M(k)$ is overbounded using the procedure found in the previous section, the resulting overbound set $\tilde{X}(k+1)$ is a union of ellipsoids with the singular vectors of each matrix of possible dynamics resulting in an expansion or contraction along the singular vectors basis.

The ellipsoids describing the set of possible state realizations are subject to a translation as to reflect the control input. Notice that the control input has no influence in the hyper-volume of the ellipsoids obtained using equation (7). However, the result of the intersection phase (see Figure 1) with the measurement set depends on the control input as it might render the intersection to be the empty set.

Due to sensor noise, the observations can be defined as a polytope $Y(k)$ posing constraints on the current state. The singular values and the associated singular vectors of the matrix defining such a polytope indicate the directions where the uncertainty is greater. Therefore, to test whether to execute the operation of computing the actual set, one can resort to intersecting the observation set $Y(k+1)$ with the ellipsoids and evaluate if the norm of the state increases over the current iteration and decide whether a *full iteration* of the SVO described in Section II is necessary.

An easy-to-compute new estimate for the norm of the state at time $k+1$ is simply obtained by solving

$$\begin{aligned} & \text{maximize} \quad \|p\| \\ & \text{subject to} \quad p^\top M(k+1)^\top M(k+1)p \leq 1 \\ & \quad \quad \quad p \in Y(k+1). \end{aligned}$$

The previous problem translates into finding an intersection of ellipsoids and then computing the point in that set with the greatest norm. The complexity is cubic in the dimension of the state as it amounts to solve a Second-Order Cone Programming (SOCP). We now summarize the

Fig. 2: Flowchart of the Self-Trigger SVO algorithm where $\mathcal{E}(k)$ and B_C are the overbounding ellipsoid at time k and the ball of radius C centered at the origin, respectively.

algorithm to self-trigger the SVO calculations (see Section II) only when the hypervolume of the overbounding set grows.

An interesting question is to determine under what conditions the set-estimate overbound converges to zero and no observations are needed. This is particularly important in fault detection, as it reduces the problem to that of checking whether the norm of the state estimate converges.

Proposition 4: Consider an LPV system

$$S : \begin{cases} x(k+1) = A(\Delta(k))x(k) + B(\Delta(k))u(k) + w(k) \\ y(k) = C(\Delta(k))x(k) \end{cases}$$

If

$$\|w(k)\| \leq (1 - \sigma_{\max}(A_0 + A_{\Delta(k)}))\|x(k)\|, \forall k \in \mathbb{N}$$

then the hyper-volume of the overbounding set is a non-increasing function.

Proof: We need to show that $\|x(k+1)\| \leq \|x(k)\|$, regardless of the disturbance, $w(k)$, which amounts to

$$\|(A_0 + A_{\Delta(k)})x(k) + w(k)\| \leq \|x(k)\|, \forall \Delta(k)$$

Using the definition of the 2-norm we get

$$\frac{\sigma_{\max}(A_0 + A_{\Delta(k)})\|x(k)\| + \|w(k)\|}{\|x(k)\|} \leq 1$$

By rearranging the terms we reach the conclusion. ■

Remark 5: Proposition 4 provides a means to ensure that, regardless of the (known) initial condition, a bounded set-valued state estimate can be computed without resorting to system output measurements. This result is shown here just for the sake of completeness, as the associated requirements can only be fulfilled if the system is stable¹. In addition,

¹In fact, it shown in [21] that stability is a necessary and sufficient condition for the assumption of Proposition 4 to be satisfied, if an arbitrarily large horizon can be considered for the SVOs.

the hyper-volume of the set thus obtained is the *worst-case* scenario, and thus is typically of no practical use.

A. Distributed Systems

The case of distributed systems is particularly relevant when considering fault detection schemes. In particular, given the formulation in (1), it is possible to accommodate a distributed system by considering that each node is represented as a state and that the sequence of actions of each node defines the Δ parameters that selects a given overall system dynamics at any time instant. A possible definition for the parameter Δ in a gossip algorithm is to have an instantiation for each edge in the graph selecting the dynamics associated with each pair of nodes communicating [20].

An important feature in distributed systems of this form is that all dynamics matrices can be rewritten as

$$A_0 + A_{\Delta_j} = P(A_0 + A_{\Delta_1})P^T, \forall j \quad (8)$$

where matrix P is a permutation matrix and j is a node different from 1. From (8) and the fact that the permutation matrix is orthogonal (i.e., $P^T P = I$), we get

$$\sigma(A_0 + A_{\Delta_j}) = \sigma(A_0 + A_{\Delta_1}).$$

Let us define M_j as the matrix generating the ellipsoid containing the set-valued state estimate, which is the set of points satisfying $\{q : q = (A_0 + A_{\Delta_j})x, x^T x \leq 1\}$. Then, the ellipsoid for any Δ_j can be defined using just Δ_1 as

$$x^T M_j x \leq 1 \Leftrightarrow (P^T x)^T M_1 P^T x \leq 1$$

which defines the ellipsoid for Δ_j at the expenses of a rotation matrix P and the ellipsoid matrix M_1 . Thus, conclusion arises that for the case of distributed systems sharing that gossip property, the set estimates overbounds will be equal ellipsoids apart from a rotation.

Remark 6: The previous argument focused on the case of distributed gossip algorithms, and found the property of the dynamics matrices being permutations of the same matrix to suffice in demonstrating that all overbounds will be the same ellipsoid up to a rotation. However, the key condition is that the dynamics matrices can be written up to an orthogonal change of basis (i.e., $\forall j, A_0 + A_{\Delta_j} = P(A_0 + A_{\Delta_1})P^T$ with $P \in O(n)$).

Theorem 7: Consider a distributed gossip algorithm with each dynamics matrix being written as $A_0 + A_{\Delta_j} = U_j S V_j^T$, where $v_{\max}^j$ is the singular vector associated with the largest singular vector and, conversely, an observation set $Y = U_y S_y V_y^T$ and $v_{\max}^y$. Then, the worst-case set-valued state estimate overbound is given by the intersection with the ellipsoid defined by Δ_j such that

$$\max_j (v_{\max}^j)^T v_{\max}^y$$

Proof: We start by noticing that we are solving the following program

$$\begin{aligned} & \text{maximize} \quad \|p\| \\ & \text{subject to} \quad p^T M_j p \leq 1 \\ & \quad \quad \quad p \in Y. \end{aligned}$$

where the $M_j = U_j S^{-2} V_j^T$ which amounts to the conversion between ellipsoid representations. Since all matrices M_j are

the same apart from a rotation, we are solving the equivalent problem

$$\begin{aligned} & \text{maximize} \quad \|p\| \\ & \text{subject to} \quad (R^\top p)^\top M_1(R^\top p) \leq 1 \\ & \quad p \in Y \\ & \quad R \in \{R_1, \dots, R_j\}. \end{aligned}$$

which, when R is unconstrained, has a closed-form solution given by $RV_y = V_1$ and the cost function evaluates to $\min(\sigma_{\max}(Y), \sigma_{\max}(A_0 + A_{\Delta_1}))$ (i.e., the maximum singular vectors align), since it is monotonically increasing with the inner product of the maximum singular vectors. Thus, the conclusion follows. ■

Theorem 7 establishes that the self-triggered technique presented in this paper is particularly effective when dealing with distributed system with a gossip feature. In such systems, the worst case scenario can be found by checking the inner product between the maximum singular vector of each possible dynamics matrix and the singular vectors of the observation set. This corresponds to having the ellipsoid aligned with $Y(k+1)$ as in Figure 1. By doing so, the computational cost associated with the combinatorial behavior of the SVOs becomes constant given that only one ellipsoid needs to be computed along with one intersection and no unions are needed. This is irrespective of the number of dynamics matrices (i.e., the number of agents).

V. MAIN PROPERTIES

In this section, we aim to present a result regarding the triggering rate for the “full” SVO computation when using the ellipsoid overbounding described previously. A node can compute the worst-case time instant for the next trigger following the results in this section, which ensures that, in between triggers, the hyper-volume of the set-valued state estimates is bounded.

The following theorem presents the value τ of time instants till the next triggers given that no measurement set is collected (i.e., transmitted to the observer). We recall that by *triggering*, it is understood the request for the computation of a “full” iteration of the standard SVO.

Theorem 8: A Self-Triggered SVO with maximum state norm at the last trigger time T given by $\|x(T)\| \leq C$, will trigger after τ time instants, where

$$\tau = \left\lceil \log_\gamma \frac{\sigma_{\min}(M(T))C}{\sqrt{n}} \right\rceil$$

with

$$\gamma = \max_{i \in \{k, \dots, k+\tau\}} \sigma_{\max}(A_0 + A_{\Delta}(i))$$

Proof: Any point $x_1 \in X(k+1)$ will satisfy $x_1 = (A_0 + A_{\Delta})x_0$ for some $x_0 \in X(k)$ and an instantiation of the uncertainties Δ . Then,

$$\begin{aligned} \|x(T+\tau)\| &= \left\| \prod_{i=0}^{\tau-1} (A_0 + A_{\Delta_{T+\tau-i}}) x(T) \right\| \\ &\leq \gamma^\tau \frac{\sqrt{n}}{\sigma_{\min}(M(T))} \end{aligned}$$

To maintain the norm bounded $\|x(T+\tau)\| \leq C$ it thus required that

$$\tau \leq \log_\gamma \frac{\sigma_{\min}(M(T))C}{\sqrt{n}}$$

Thus, leading to the conclusion. ■

The previous result is of particular interest for the case where the module running the observer and the plant itself are connected using a network and where communication is expensive. Under these circumstances, the worst-case triggering decision must be made based solely on the model dynamics and not on the observation set.

VI. SIMULATION RESULTS

In this section, we show illustrative simulation results regarding the use of self-triggered SVOs. In Section III, we assumed a particular structure for the matrix defining the polytopes. However, the aim of this paper does not concern devising an algorithm and, during simulation, we performed a simple method by duplicating the original constraints on the state, to its symmetric counterpart. The new set is thus obtained by computing the convex hull of these two sets. Such procedure introduces conservatism and is a future research topic. Focus is given on how the triggering time reduces the computational cost associated with calculating the set-valued estimates.

We consider a simplified wind turbine problem model where the continuous-time state dynamics are given by

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\chi\omega \end{bmatrix}$$

The system is discretized using a sampling period of 0.01s and the whole simulation lasts 100 iterations. We assume that the measurement noise and the exogenous disturbances are bounded. The control input is set to be a constant signal value for the sake of simplicity. The objective is to simulate how the standard set-valued state estimates evolve in comparison with those from the self-triggered approach, in a disadvantageous scenario, namely considering that the generated polytope is not centered at the origin, which introduces conservatism in the solution.

Figure 3 depicts a typical run of the algorithm with the triggering times reducing the volume of the estimated set. Both the standard and self-triggering approaches are shown. In the self-triggering approach, the volume increments depend on the singular values of the dynamics matrix, as well as on the bounds for the disturbance and input signals. The triggering times become more frequent as the polytope resulting from the standard SVO framework becomes less centered at the origin. Moreover, at a given point of the simulation, a trigger occurs every other time step due to the conservatism. Thus, future research attention will focus on how to enforce the symmetry with less conservatism.

Figure 4 depicts the execution time of each iteration for both cases. The key observation is that when we get a trigger, the execution time is very similar between the two cases. However, we gain from refraining the computation in each time instant. The peaks at the beginning are not very meaningful in the sense that they correspond to the first iteration, where all the data structures need to be allocated.

Fig. 3: Volume of the set-valued estimates when using the standard (blue line) and self-triggering (green line) approaches.

Fig. 4: Computing elapsed time for each iteration where the blue line is for the SVO and the green line for the Self-triggering approach.

VII. CONCLUSIONS

In this paper, the problem of high computational demand from the use of SVOs in the fault detection domain is addressed using an overapproximation technique, which only triggers the standard SVO computations when the volume of the set-valued state estimation is increasing. The method uses ellipsoids instead of the original polytopes, which inherits two main advantages, namely i) the number of rows in the matrices defining ellipsoids is in general much smaller; and ii) to compute linear transformations we can use the maximum singular value of the matrix defining the ellipsoid.

The proposed self-trigger method allows the observer to use the approximations whenever the hyper-volume of the set is converging, and compute the actual (or an improved approximation of the) polytope whenever such measure is above the previous triggering time. By doing so, the compu-

tations can be performed only at triggering times, avoiding the complex and heavy computations of the standard SVOs. For the particular case of distributed gossip algorithms, it is shown that the singular vectors are identical up to a rotation matrix, which enables the estimate update to be performed using only the worst-case singular vector of the measurement and prediction sets. The problem of computing the trigger rate and the time to the next trigger is also investigated.

REFERENCES

- [1] R. Motwani and P. Raghavan, *Algorithms and Theory of Computation Handbook*. Chapman & Hall/CRC, 2010, ch. Randomized Algorithms, pp. 12–12.
- [2] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, “Randomized gossip algorithms,” *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2508 – 2530, Jun. 2006.
- [3] A. S. Willsky, “A survey of design methods for failure detection in dynamic systems,” *Automatica*, vol. 12, no. 6, pp. 601 – 611, 1976.
- [4] I. Hwang, S. Kim, Y. Kim, and C. Seah, “A survey of fault detection, isolation, and reconfiguration methods,” *Control Systems Technology, IEEE Transactions on*, vol. 18, no. 3, pp. 636 –653, may 2010.
- [5] J. Bokor and G. Balas, “Detection filter design for LPV systems – a geometric approach,” *Automatica*, vol. 40, pp. 511–518, 2004.
- [6] S. Narasimhan, P. Vachhani, and R. Rengaswamy, “New nonlinear residual feedback observer for fault diagnosis in nonlinear systems,” *Automatica*, vol. 44, pp. 2222–2229, 2008.
- [7] P. Rosa and C. Silvestre, “Fault detection and isolation of LPV systems using set-valued observers: An application to a fixed-wing aircraft,” *Control Engineering Practice*, vol. 21, no. 3, pp. 242 – 252, 2013.
- [8] H. Witsenhausen, “Sets of possible states of linear systems given perturbed observations,” *IEEE Transactions on Automatic Control*, vol. 13, no. 5, pp. 556 – 558, oct 1968.
- [9] F. Schweppe, “Recursive state estimation: Unknown but bounded errors and system inputs,” *IEEE Transactions on Automatic Control*, vol. 13, no. 1, pp. 22 – 28, feb 1968.
- [10] F. Schweppe, *Uncertain Dynamic Systems*. Prentice-Hall, 1973.
- [11] M. Milanese and A. Vicino, “Optimal estimation theory for dynamic systems with set membership uncertainty: An overview,” *Automatica*, vol. 27, no. 6, pp. 997 – 1009, 1991.
- [12] D. Bertsekas and I. Rhodes, “Recursive state estimation for a set-membership description of uncertainty,” *IEEE Transactions on Automatic Control*, vol. 16, no. 2, pp. 117 – 128, apr 1971.
- [13] T. Alamo, J. Bravo, and E. Camacho, “Guaranteed state estimation by zonotopes,” *Automatica*, vol. 41, no. 6, pp. 1035 – 1043, 2005.
- [14] R. E. Moore, *Interval analysis*, ser. Prentice-Hall series in automatic computation. Englewood Cliffs, NJ: Prentice-Hall, 1966.
- [15] P. Rosa, “Multiple-Model Adaptive Control of Uncertain LPV Systems,” Ph.D. dissertation, Technical University of Lisbon, Lisbon, Portugal, 2011.
- [16] P. Rosa and C. Silvestre, “On the distinguishability of discrete linear time-invariant dynamic systems,” in *50th IEEE Conference on Decision and Control*, December 2011.
- [17] J. Shamma and K.-Y. Tu, “Set-valued observers and optimal disturbance rejection,” *IEEE Transactions on Automatic Control*, vol. 44, no. 2, pp. 253 –264, feb 1999.
- [18] S. Keerthi and E. Gilbert, “Computation of minimum-time feedback control laws for discrete-time systems with state-control constraints,” *IEEE Transactions on Automatic Control*, vol. 32, no. 5, pp. 432 – 435, may 1987.
- [19] P. Rosa, C. Silvestre, J. Shamma, and M. Athans, “Fault detection and isolation of LTV systems using set-valued observers,” *49th IEEE Conference on Decision and Control*, pp. 768–773, December 2010, Atlanta, Georgia, USA.
- [20] D. Silvestre, P. Rosa, R. Cunha, J. P. Hespanha, and C. Silvestre, “Gossip average consensus in a byzantine environment using stochastic set-valued observers,” in *52nd IEEE Conference on Decision and Control*, 2013, Florence, Italy.
- [21] P. Rosa, C. Silvestre, and M. Athans, “Model falsification using set-valued observers for a class of discrete-time dynamic systems: a coprime factorization approach,” *International Journal of Robust and Nonlinear Control*, vol. 24, no. 17, pp. 2928–2942, 2014.