Dynamic Programming Lecture #12

Outline:

- Example: Machine repair
- Example: Hypothesis testing

Set-Up

- System:

$$x_{k+1} = f_k(x_k, u_k, w_k)$$

- Observations:

$$z_0 = h_0(x_0, v_0)$$
$$z_k = h_k(x_k, u_{k-1}, v_k)$$

  - Initial condition is random.
  - Measurement disturbance (noise) v_k is random, but can depend on $(x_{k-1}, u_{k-1}, w_{k-1})$.

- Solution: Apply DP to:

$$X_{k+1} = \Phi_k(X_k, u_k, z_{k+1})$$

Machine Repair

- Machine state: $= \begin{cases} 1 & \text{up} \\ 0 & \text{down} \end{cases}$.
- Disturbance: Machine failure.
- Control actions: $= \begin{cases} C & \text{continue (do nothing)} \\ S & \text{stop \& repair if needed} \end{cases}$.
- System probabilities: If up, stay up with probability $2/3$ & fail with probability $1/3$.
- Measurement: State of machine.
- Measurement noise: Correct diagnosis with probability $3/4$ & false diagnosis with probability $1/4$
- Stage costs:

$$g(x,u) = \begin{cases} 2 & \text{continue with machine down} \\ 0 & \text{continue with machine up} \\ 1 & \text{stop \& repair} \end{cases}$$

- Let $p_k =$ probability machine is down at time k given information up to time k. (Then $(1 - p_k)$ is probability machine is up at time k.)
- What is p_1 as a function of p_0, u_0, & z_1? i.e., derive

$$X_1 = \Phi(X_0, u_0, z_1)$$

Machine Repair, cont (2)

- In each case, compute
$$\frac{P(\text{down} \ \ \& \ u_0, z_1)}{P(u_0, z_1)}$$

- $u_0 = S \ \& \ z_1 = U$:
$$\begin{cases} (1/3)(1/4) & \text{(fail)(false)} \\ (2/3)(3/4) & \text{(no fail)(true)} \end{cases} \Rightarrow \frac{1/12}{1/12 + 6/12} = 1/7$$

- $u_0 = S \ \& \ z_1 = D$:
$$\begin{cases} (1/3)(3/4) & \text{(fail)(true)} \\ (2/3)(1/4) & \text{(no fail)(false)} \end{cases} \Rightarrow \frac{1/4}{1/4 + 2/12} = 3/5$$

- $u_0 = C \ \& \ z_1 = U$:
$$\begin{cases} (1-p_0)(1/3)(1/4) & \text{(was up)(fail)(false)} \\ p_0(1/4) & \text{(was down)(false)} \\ (1-p_0)(2/3)(3/4) & \text{(was up)(no fail)(true)} \end{cases} \Rightarrow \frac{1+2p_0}{7-4p_0} = \frac{1+2}{1+2+3}$$

- $u_0 = C \ \& \ z_1 = D$:
$$\begin{cases} (1-p_0)(1/3)(3/4) & \text{(was up)(fail)(true)} \\ (1-p_0)(2/3)(1/4) & \text{(was up)(no fail)(false)} \\ p_0(3/4) & \text{(was down)(true)} \end{cases} \Rightarrow \frac{3+6p_0}{5+4p_0} = \frac{1+3}{1+2+3}$$

- In general: $p^+ = \Phi(p, u, z^+)$

Machine Repair, cont (3)

- Now consider a horizon $[0, 2]$ with no terminal cost.

$$J_2(X_2) = 0$$

$$J_1(X_1) = \min_u E\{g(x,u)\} = \begin{cases} 2p_1 + 0 * (1 - p_1) = 2p_1 & \text{for } u_1 = C \\ 1 & \text{for } u_1 = S \end{cases}$$

$$\mu_1(p_1) = \begin{cases} C & \text{if } 2p_1 < 1 \\ S & \text{otherwise} \end{cases}$$

$$J_1(p_1) = \min\{2p_1, 1\}$$

$$J_0(p_0) = \min_u E\{g(x,u) + J_1(\Phi(X_0, u_o, z_1))\}$$

  – Case $u_0 = S$:

$$\begin{aligned} J_0(p_0) &= 1 + E\{J_1(\Phi(X_0, S, z_1))\} \\ &= 1 + J_1(1/7)P(z_1 = U) + J_1(3/5)P(z_1 = D) \\ &= 1 + J_1(1/7)(7/12) + J_1(3/5)(5/12) \\ &= 1 + (2 * 1/7)(7/12) + 1(5/12) \end{aligned}$$

  – Case $u_0 = C$:

$$\begin{aligned} J_0(p_0) &= 0 * (1 - p_0) + 2p_0 + J_1(\frac{1 + 2p_0}{7 - 4p_0})P(z_1 = U) + J_1(\frac{3 + 6p_0}{5 + 4p_0})P(z_1 = D) \\ &= \text{etc} \end{aligned}$$

- Final step: Compare $J_0(p_0)$ for $u_0 = C$ vs $u_0 = S$.

Sequential Hypothesis Testing

- Set-up: Measure $\{z_0, z_1, \ldots, z_{N-1}\}$, i.e., $z_i \in Z$.
- Objective: Are z's generated by distribution f_0 or f_1?
- Decision: At time k
  - Stop observing and conclude f_0 or f_1.
  - Take and additional observation at a cost C.
- Losses: If we choose to stop
  - Cost $= L_0$ if f_0 is chosen and is wrong.
  - Cost $= L_1$ if f_1 is chosen and is wrong.
- State equations: $x_{k+1} = x_k$, where state is either 0 or 1.
- Initial condition: $P(x_0 = 0) = p$

Hypothesis Testing cont, (2)

- Notation: Let $p_k = P(x_k = 0|I_k)$.
- $p_0 =$?:

$$\begin{aligned} P(x_0 = 0|z_0 = z^*) &= \frac{P(x_0 = 0 \ \& \ z_0 = z^*)}{P(z_0 = z^*)} \\ &= \frac{pf_0(z^*)}{pf_0(z^*) + (1-p)f_1(z^*)} \end{aligned}$$

So

$$p_0 = \frac{pf_0(z_0)}{pf_0(z_0) + (1-p)f_1(z_0)}$$

- $p_1 =$?: Same analysis

$$p_{k+1} = \frac{p_k f_0(z_{k+1})}{p_k f_0(z_{k+1}) + (1-p_k)f_1(z_{k+1})}$$

Probability Propagation Illustration

- Suppose

$$Z = \{A, B\}$$
$$f_0 \simeq \{0.99, 0.01\}$$
$$f_1 \simeq \{0.01, 0.99\}$$

- Suppose *a priori* $p = 1/3$:
  - Case measure $z = A$:

$$p_0^+ = \frac{(1/3)(0.99)}{(1/3)(0.99) + (2/3)(0.01)} = 0.98$$

  - Case measure $z = B$:

$$p_0^+ = \frac{(1/3)(0.01)}{(1/3)(0.01) + (2/3)(0.99)} = 0.005$$

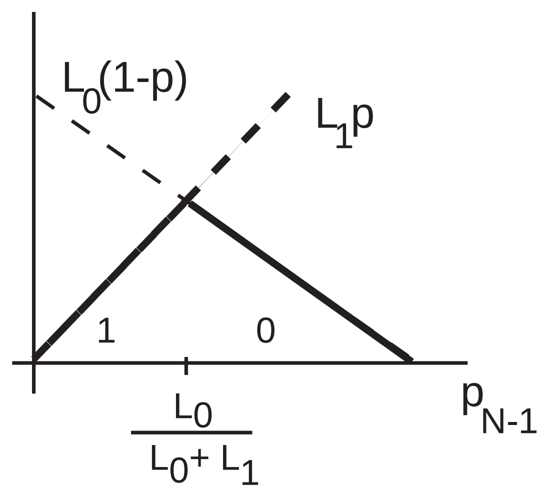
Hypothesis Testing, cont (3)

- Now apply DP

$$J_N(p_N) = \min_{\text{choose } 0 \text{ or } 1} E\{\text{cost of choice}\}$$

  - Choose 0: $L_0(1-P_N)$
  - Chose 1: $L_1 P_N$

- $J_N(p_N) = \min\{L_0(1-P_N), L_1 P_N\}$



- If $L_0 \gg L_1 \Rightarrow$ always choose 1 unless VERY confident.

Hypothesis Testing, cont (4)

- DP recursions:

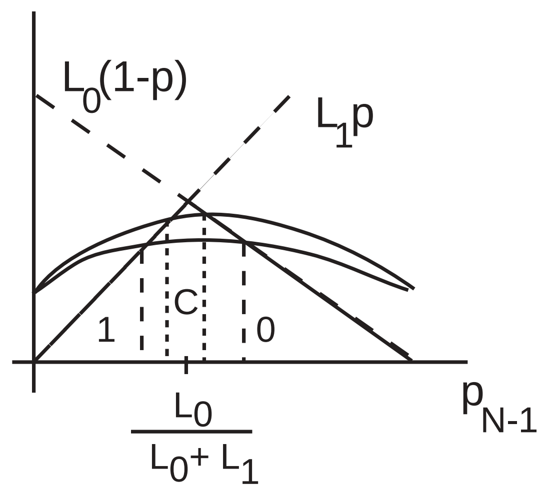
$$J_k(p_k) = \min \begin{cases} (1-p_k)L_0 \\ p_k L_1 \\ C + A_k(p_k) \end{cases}$$

where

$$\begin{aligned}
A_k(p_k) &= E\left\{ J_{k+1}\left(\tfrac{p_k f_0(z_{k+1})}{p_k f_0(z_{k+1}) + (1-p_k) f_1(z_{k+1})} \right) \right\} \\
&= \left(\textstyle\sum_i f_0(z^i) J_{k+1}\left(\tfrac{p f_0(z^i)}{p f_0(z^i) + (1-p) f_1(z^i)} \right) \right) p \\
&\quad + \left(\textstyle\sum_i f_1(z^i) J_{k+1}\left(\tfrac{p f_0(z^i)}{p f_0(z^i) + (1-p) f_1(z^i)} \right) \right) (1-p) \\
&= \sum_i \left(p f_0(z^i) + (1-p) f_1(z^i) \right) J_{k+1}\left(\tfrac{p f_0(z^i)}{p f_0(z^i) + (1-p) f_1(z^i)} \right)
\end{aligned}$$

Hypothesis Testing, cont (5)

- Facts:
  - $A_k(0) = A_k(1) = 0.$
  - $A_{k-1}(p) \leq A_k(p)$ by monotonicity.
  - $A_k(p)$ concave



- Structure of optimal thresholds:
  - Pick 1 if in left region.
  - Pick 0 if in right region.
  - Pick C if in middle.
  - Thresholds converge as horizon length approaches infinity.