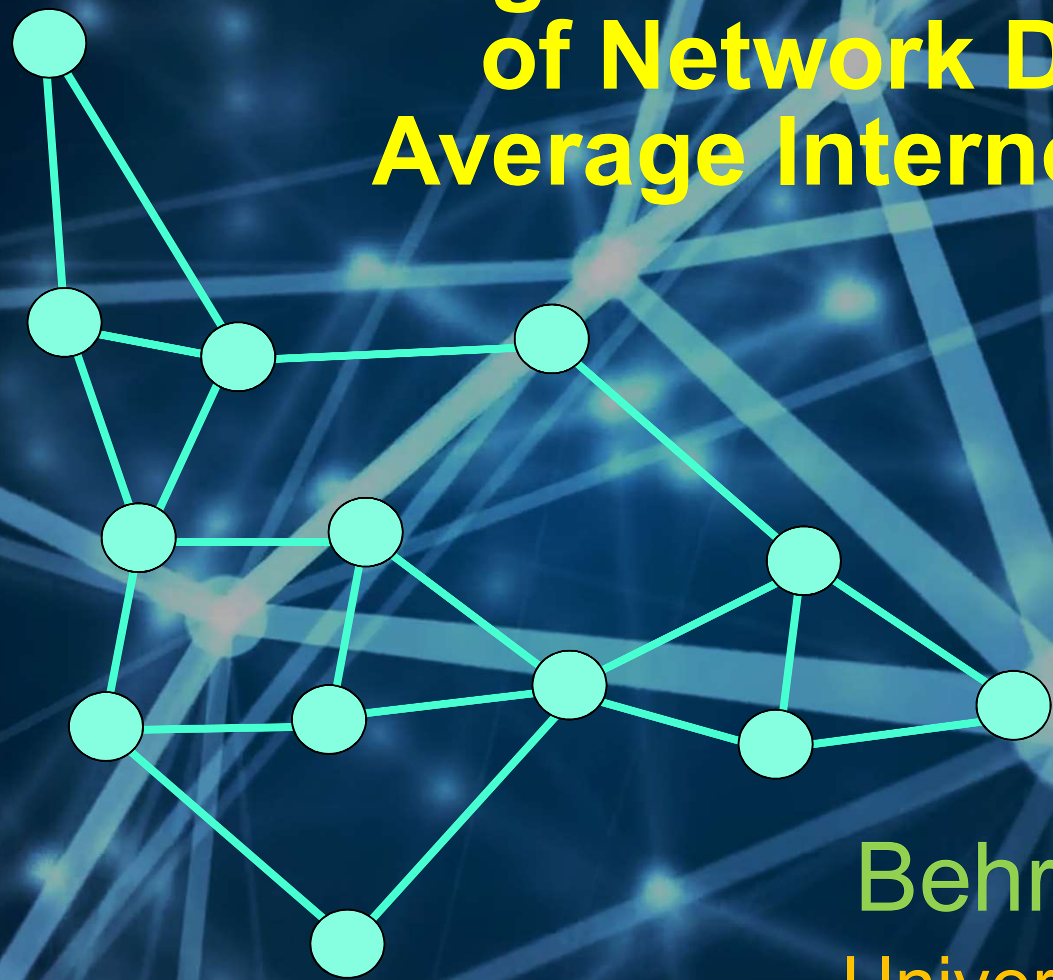
Tight Bounds on the Ratio of Network Diameter to Average Internode Distance



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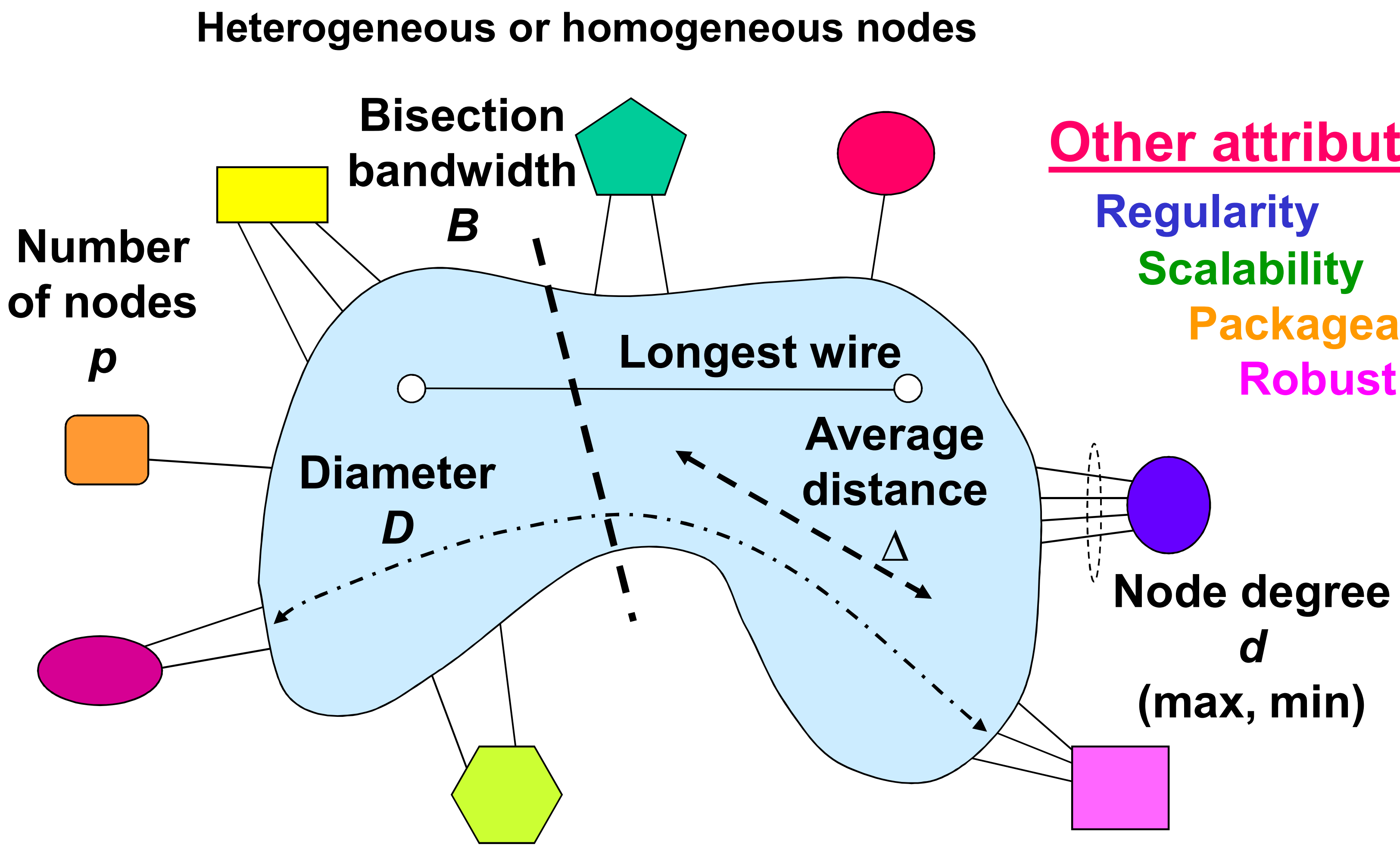
About This Presentation

This slide show was first developed in fall of 2018 for a November 2018 talk at IEEE IEMCON (Information Technology, Electronics & Mobile Communication Conf.), University of British Columbia, Vancouver, BC, Canada.

Edition	Released	Revised	Revised	Revised
First	Fall 2018			

Network Attributes



Other attributes:

- Regularity
- Scalability
- Packageability
- Robustness

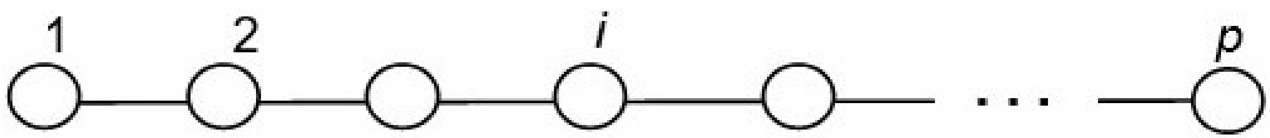
Distances in Path and Mesh Networks

$$D_{p\text{-path}} = p - 1$$

$$\Delta_{p\text{-path}} = (1/p^2)\Sigma_{0\le j\le p-1}[\Sigma_{0\le i\le j}(j - i) + \Sigma_{j\le i\le p-1}(i - j)]$$

$$\Delta_{p\text{-path}} = (1/p^2)\Sigma_{0\le j\le p-1}[j(j + 1) - j(j + 1)/2$$

$$+ (p - j)(p - 1 + j)/2 - j(p - j)] = (1/3)(p - 1/p)$$



$$D_{q\text{D-mesh}} = \Sigma_{1\le i\le q} n_i - q$$

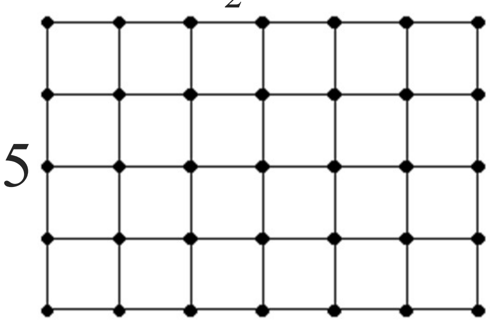
$$\Delta_{q\text{D-mesh}} = (1/3)[\Sigma_{1\le i\le q}(n_i - 1/n_i)]$$

$$D_{p\text{-path}} \,/\, \Delta_{p\text{-path}} \cong 3$$

$$D_{q\text{D-mesh}} \,/\, \Delta_{q\text{D-mesh}} \cong 3$$

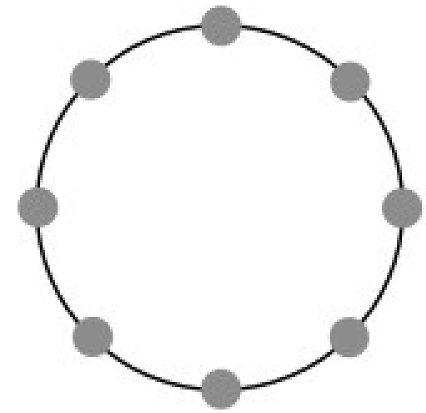
$n_2 = 7$

$n_1 = 5$



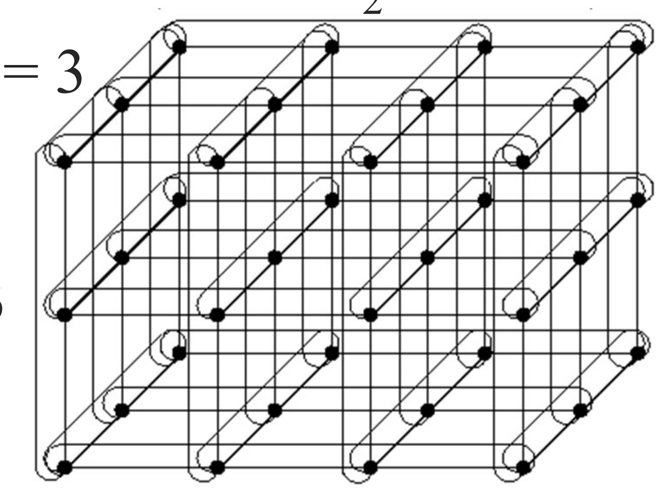
Distances in Ring and Torus Networks

$$D_{p\text{-ring}} = (1/2)[p - (p \bmod 2)/p]$$
$$\Delta_{p\text{-ring}} = (1/4)[p - (p \bmod 2)/p]$$
$$D_{q\text{D-torus}} = (1/2)\Sigma_{1 \le i \le q}[n_i - (n_i \bmod 2)/n_i]$$
$$\Delta_{q\text{D-torus}} = (1/4)\Sigma_{1 \le i \le q}[n_i - (n_i \bmod 2)/n_i]$$



Alternative formula: $D_{p\text{-ring}} = \lceil (p - 1)/2 \rceil$

$$D_{p\text{-ring}} / \Delta_{p\text{-ring}} = 2$$
$$D_{q\text{D-torus}} / \Delta_{q\text{D-torus}} = 2$$



$n_2 = 4$

$n_3 = 3$

$n_1 = 3$

Distances in Complete Binary Trees (1)

$D_{\text{binary-tree}} = 2l - 2 = 2 \log_2 m - 2$

[Let $m = 2^l$; T_m has $2^l - 1$ nodes]

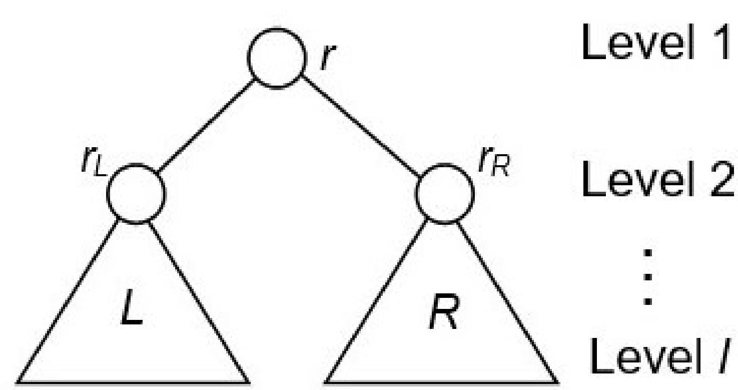
$\sigma(T_m) = 1\times 2^1 + 2\times 2^2 + \ldots + (l - 1)\times 2^{l-1} = (l - 2)2^l + 2$

$= m \log_2 m - 2m + 2$

$S(L, L) = S(R, R) = S(T_{m/2})$

$S(r, L) = S(r, R) = S(L, r) = S(R, r)$

$= m/2 - 1 + \sigma(m/2)$



$S(L,R) = S(R,L)$

$= (m/2–1)^2[2 + 2\sigma(m/2)/(m/2–1)]$

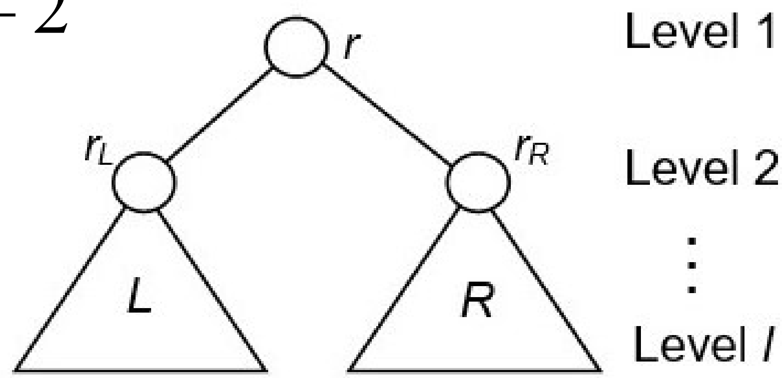
$= (m - 2)\sigma(m/2) + (m - 2)^2/2$

Distances in Complete Binary Trees (2)

$$S(T_m) = 2S(L, L) + 4S(r, L) + 2S(L, R)$$
$$= 2S(T_{m/2}) + m^2 \log_2 m - 2m^2 + 2m$$
$$= 2m^2 \log_2 m - 6m^2 + 2m \log_2 m + 6m$$

$$\Delta(T_m) = (2m^2 \log_2 m - 6m^2 + 2m \log_2 m + 6m)/(m - 1)^2$$
$$= \underbrace{2 \log_2 m - 6}_{\text{Asymptotic value}} + 2(3m \log_2 m - 3m - \log_2 m + 3)/(m - 1)^2$$

Recall $D(T_m) = 2l - 2 = 2 \log_2 m - 2$

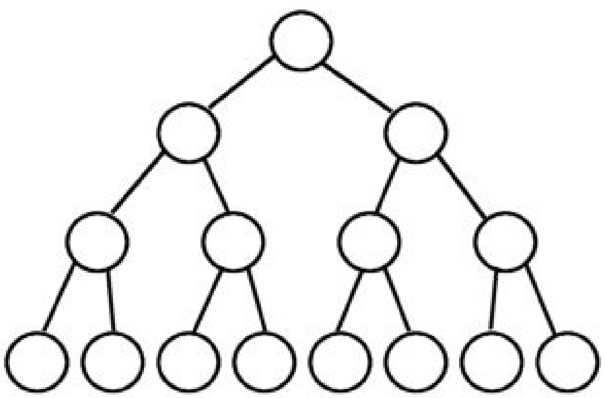


$$\lim_{m \to \infty} \Delta(T_m) = D(T_m) - 4$$
$$\lim_{m \to \infty} D(T_m) / \Delta(T_m) = 1$$

Incomplete and Balanced Binary Trees

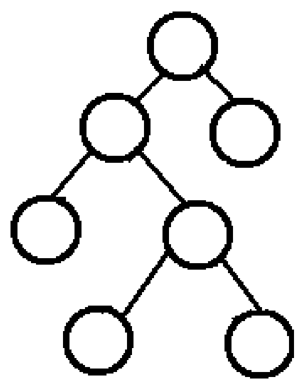
Complete binary tree: All $2^{l-1} = (p + 1)/2$ leaves are at level l
Incomplete binary tree: There are leaves in 2 or more levels
Balanced binary tree: Leaves are at levels l and $l - 1$
Complete binary tree: All leaves are at level l

$p = 2^l - 1$



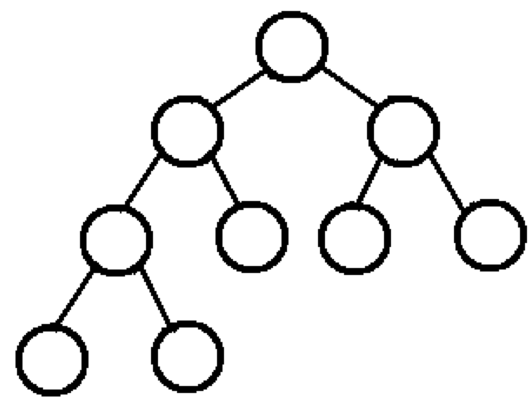
Complete

$p < 2^l - 1$



Incomplete

$2^{l-1} - 1 < p < 2^l - 1$



Balanced

Distances in Balanced Binary Trees

Theorem 1: In an incomplete binary tree with more than one incomplete level, removing a node from an incomplete level k and adding a node to an incomplete level $k - j$ ($j > 0$) does not increase the diameter and always reduces the average internode distance. ■

Theorem 2: In a balanced binary tree, with the final level l containing missing nodes in both subtrees, removing a node from a side with equal or fewer nodes and adding a node to the other side decreases the average internode distance, with no increase in diameter. ■

Extremes in Distance Ratio Bounds

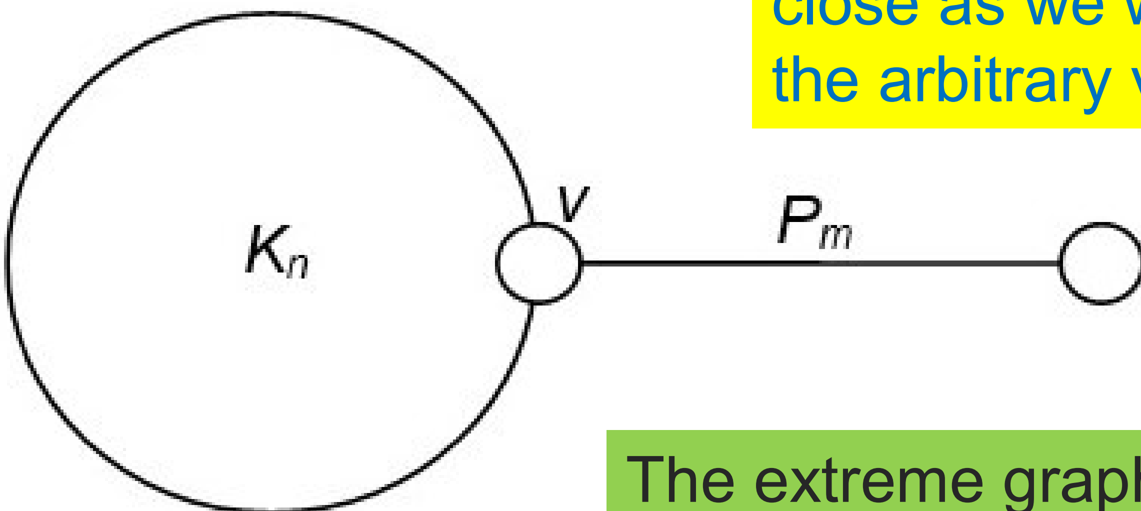
$D(G) = m$

$\Delta(G) = [n^2 + m(m^2 - 1)/3 + 2(n-1)(2+3+\ldots+m)]/(n+m-1)^2$

$\lim_{n\to\infty} \Delta(G) = 1$

$\lim_{n\to\infty} D(T_m) / \Delta(T_m) = m$

So, we can make the D/Δ ratio as close as we want to the arbitrary value m



The extreme graph G

Ratio Bounds in Symmetric Networks

Theorem 3: Given a node-symmetric network with node degree d, diameter D, and average internode distance Δ, we have $D/2 \le \Delta \le D$. ■

Proof outline: Consider a node X and a diametrically opposite node to it, Y. Let there be d nodes that are distance-1 to X (its immediate neighbors). By node-symmetry, Y also has d distance-1 nodes. The latter nodes are at least distance $D - 1$ to X. So, the average distance from X to the two set of nodes (neighbors of X and Y) is at least $D/2$. This process can be repeated for distance-2, distance-3, … nodes, until done. ■

Some Practical Implications

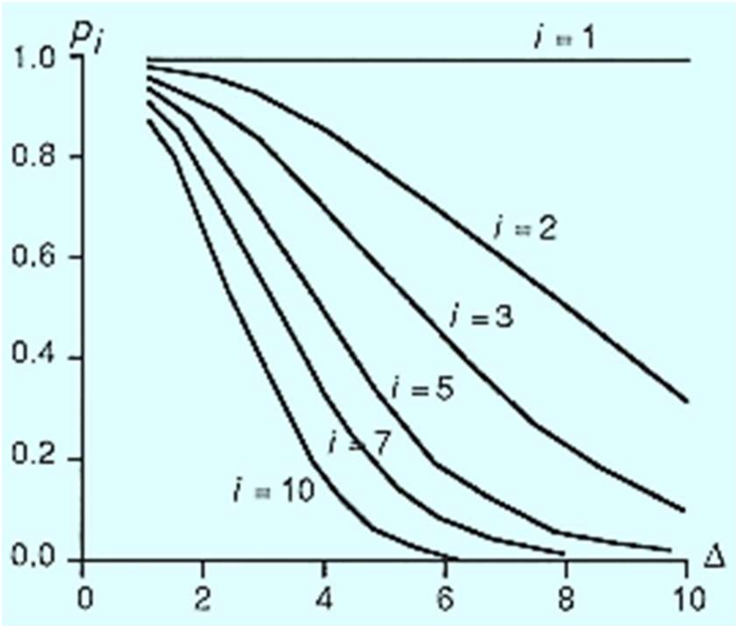
D and Δ are important network parameters

Can't judge a network merely on the basis of its aggregate bandwidth *Bw*

Consider a 100-link network, with *Bw* = 100*b*

Probability of being able to establish an *i*th random routing path of length Δ in the network is

$$p_i = \binom{C-(i-1)\Delta}{\Delta} \Big/ \binom{C}{\Delta}$$



Conclusions and Future Work

Calculating average distance avoidable in many cases

Ratio of diameter to average internode distance is:

Unbounded in worst-case (impractical extremes)

D/Δ

Between 1 and 2 in symmetric networks

Fairly small in other practical cases

Very close to 1 for trees

Future work and practical impact

Tighten the bounds for special classes of networks

Study pertinent bounds for Cayley graphs

Simulate in detail effects of D and Δ

Derive exact Δ for more networks

Routing-based D and Δ

Questions or Comments?

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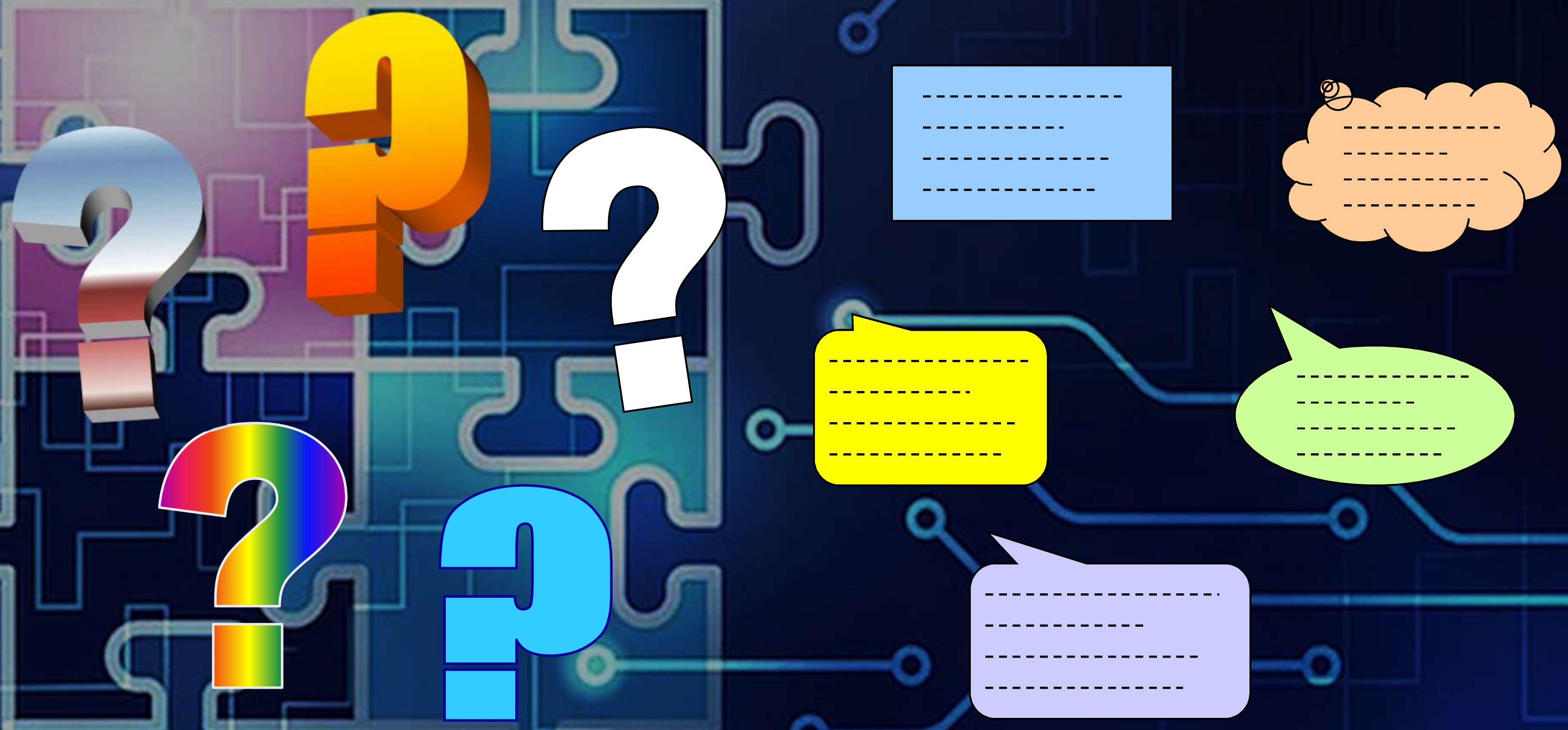

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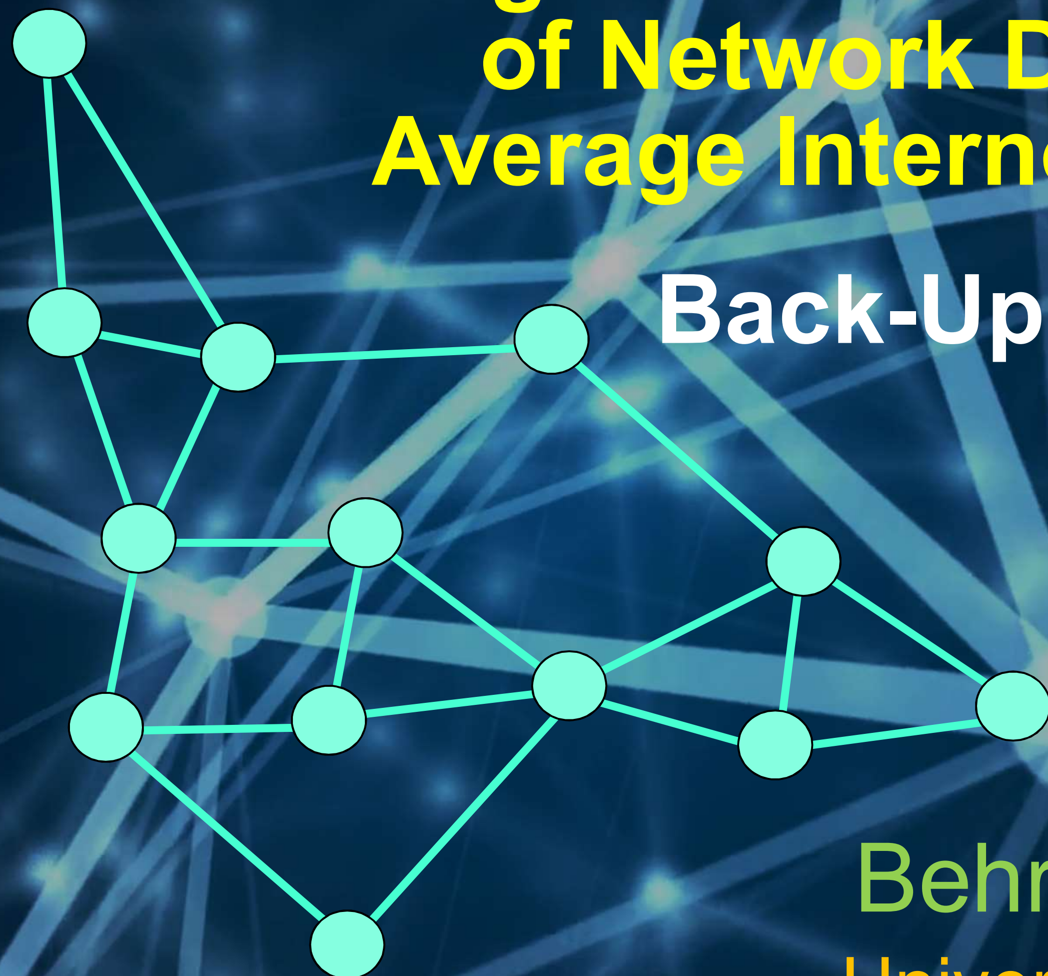
Back-Up Slides



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Effect of Δ in Establishing Routing Paths

Probability of being able to establish an *i*th random routing path of length Δ in a 100-link network

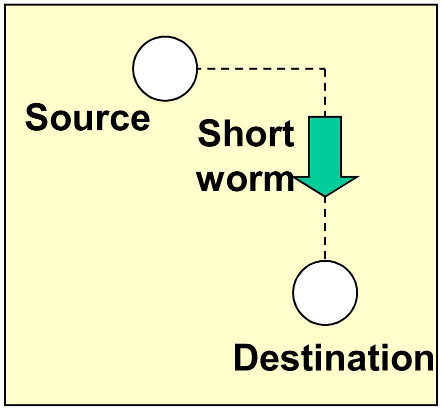
Δ↓ *i*→	2	3	5	7	10
1	0.990	0.980	0.960	0.940	0.910
2	0.960	0.921	0.846	0.773	0.671
3	0.912	0.829	0.679	0.548	0.385
4	0.847	0.713	0.492	0.327	0.162
5	0.770	0.584	0.319	0.161	0.046
6	0.683	0.455	0.183	0.063	0.008
7	0.592	0.336	0.092	0.019	0.001
8	0.500	0.234	0.040	0.004	0.000
9	0.395	0.154	0.014	0.001	0.000
10	0.310	0.095	0.004	0.000	0.000

Routing with Wormhole Switching

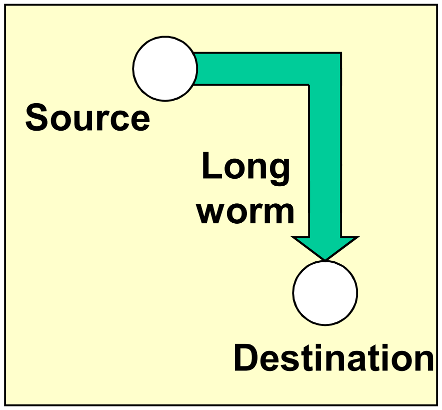
Average internode distance Δ is an indicator of performance

Δ is closely related to the diameter D

For symmetric nets: $D/2 \le \Delta \le D$



Short worms: hop distance clearly dictates the message latency



Long worms: latency is insensitive to hop distance, but tied up links and waste due to dropped or deadlocked messages rise with hop distance