Hybrid Digital-Analog Number Representation in Computing and in Nature

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About This Presentation

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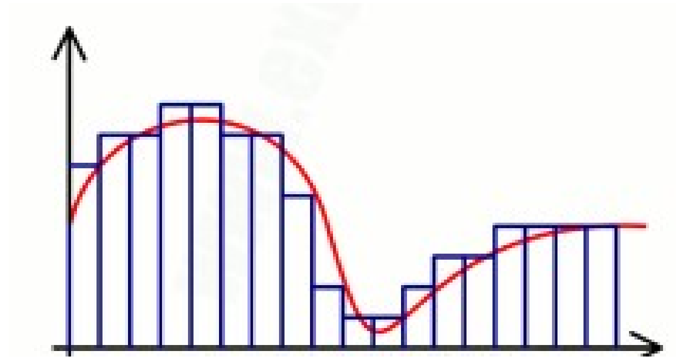
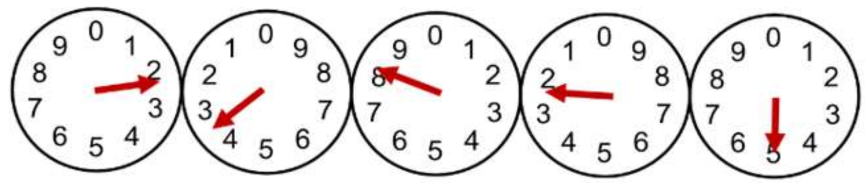
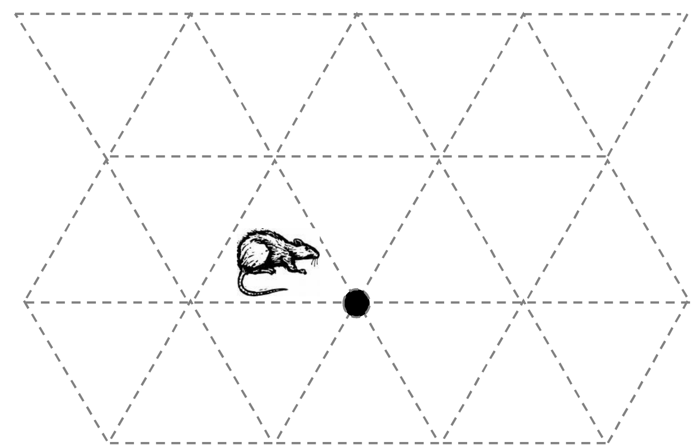
Edition	Released	Revised	Revised	Revised
First	Nov. 2020	May 2022		

Outline

- **Introduction and Background**
  - Why analog is cool again
  - Residue number system (RNS)
  - The brain and numeracy
- **Hybrid Numbers and Arithmetic**
  - Historical perspective
  - Some hybrid representations
  - Nobel Prize in medicine, 2014
- **Continuous-Digits RNS**
  - CD-RNS models rat's navigation
  - Different from discrete RNS
  - Dynamic range and precision
  - Other properties and challenges
- **Conclusions and Future Work**





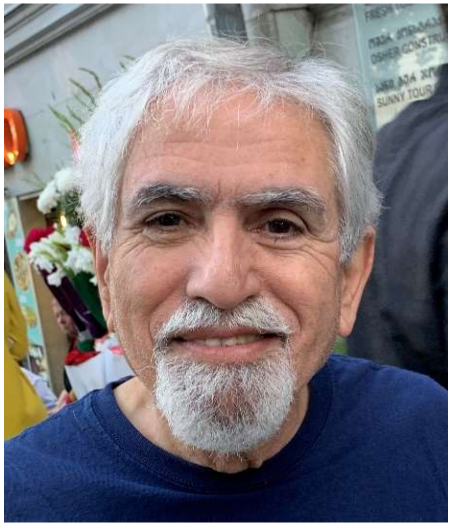


Abstract

The discovery that mammals use a multi-modular method akin to residue number system (RNS), but with continuous residues or digits, to encode position information led to the award of the 2014 Nobel Prize in Medicine. After a brief review of the evidence in support of this hypothesis, and how it relates to RNS, I discuss the properties of continuous-digit RNS, and discuss results on the dynamic range, representational accuracy, and factors affecting the choice of the moduli, which are themselves real numbers. I then take a step back and briefly explore hybrid digital-analog number representations and their robustness and noise-immunity advantages more generally. I conclude with suggestions for further research on important open problems in the domain of hybrid digital-analog number representation and processing.

Speaker’s Brief Technical Bio

Behrooz Parhami (PhD, UCLA 1973) is Professor of Electrical and Computer Engineering, and former Associate Dean for Academic Personnel, College of Engineering, at University of California, Santa Barbara, where he teaches and does research in the field of computer architecture: more specifically, in computer arithmetic, parallel processing, and dependable computing.



A Life Fellow of IEEE, a Fellow of IET and British Computer Society, and recipient of several other awards (including a most-cited paper award from *J. Parallel & Distributed Computing*), he has written six textbooks and more than 300 peer-reviewed technical papers. Professionally, he serves on journal editorial boards (including for 3 different *IEEE Transactions*) and conference program committees, and he is also active in technical consulting.

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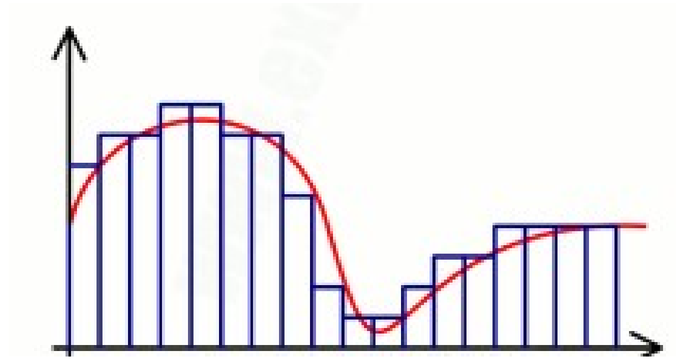
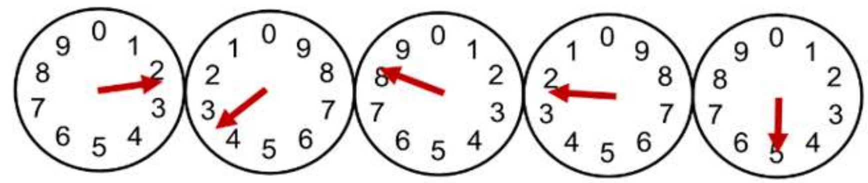
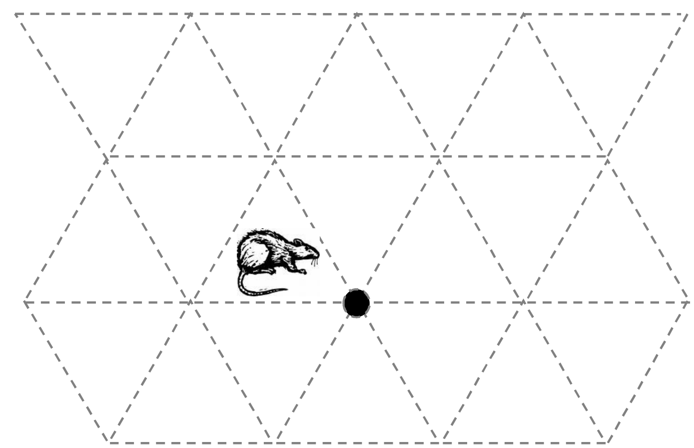






How Looking at Nature Helps my Research

Parallel processing

Parallelism used extensively in human brain and other natural systems

Dependable (fault-tolerant) computing

The self-healing amphibian axolotl can regenerate a near-perfect replica of almost any body part it loses

Computer arithmetic

My subject area today: Use of residue representation in rat's navigational system

Analog Computation Is Back!

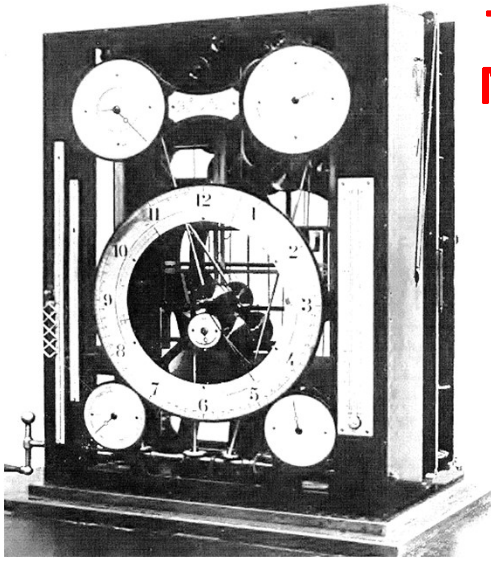
Digital data has been replacing analog data for years:
From 1986 to 2007, share of digital stored data went from near-zero to 90%; a decade later, digital was fully dominant

We now struggle with complexity and energy limitations:
The solution seems to be approximate & analog computing

Hybrid digital/analog can give us the best of both worlds:
Digital's higher latency and power used only when needed

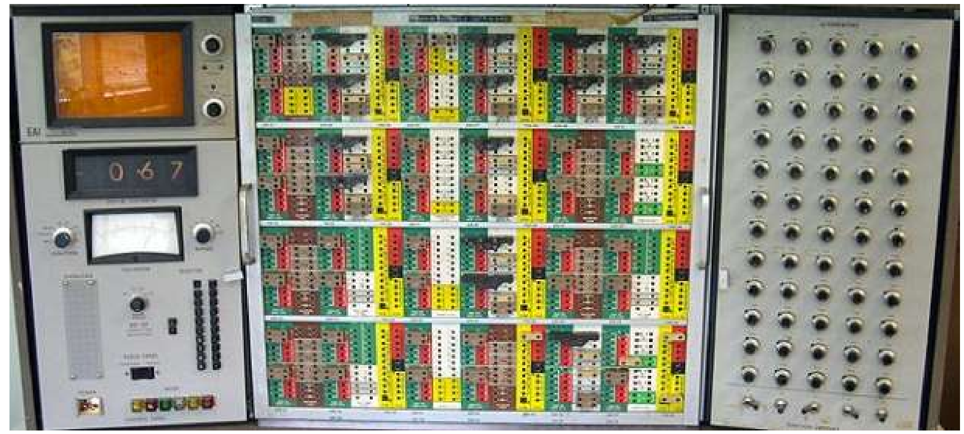
Hence, multi-resolution representations are desirable:
Combine fast, efficient low-precision computation with slower, energy-intensive high-precision computation

Analog Computers as Simulators



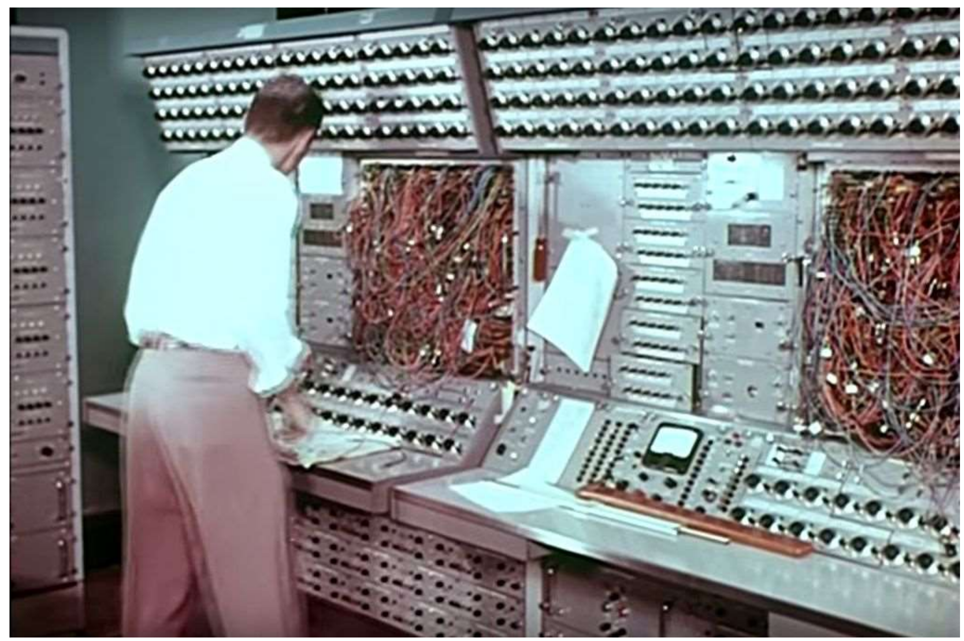
Tide-Predicting Machine, 1880s
(Wikipedia)

Pace TR-48: Programming via Plugboard



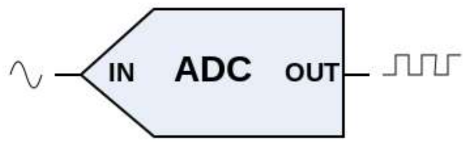
Telefunken RAT 700/2
(Computer History Museum)

X-15 Orbiter Simulator
(Wikipedia)

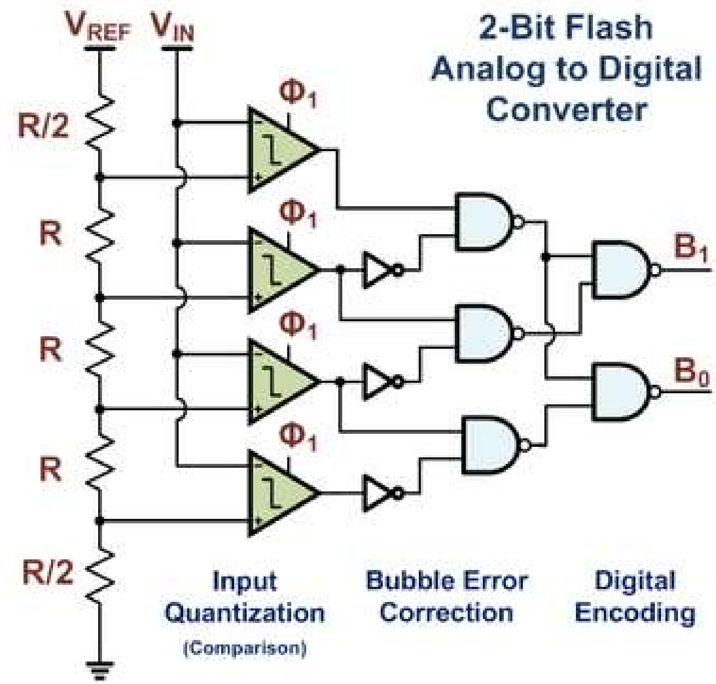


A/D and D/A Conversion

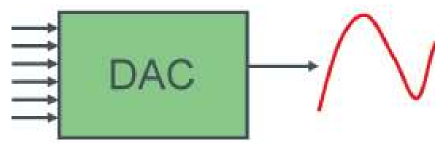
- Analog-to-Digital



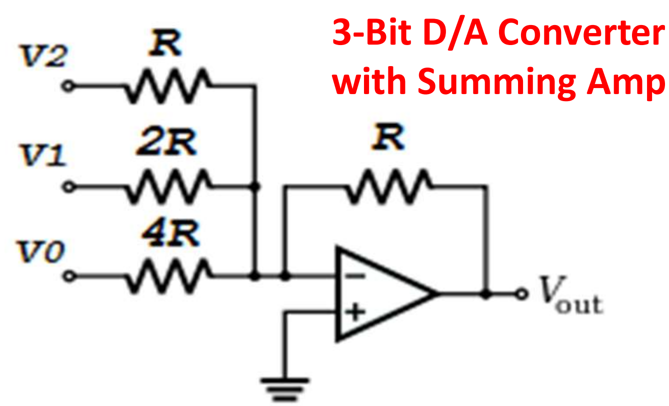
  - Sound from microphone
  - IoT instruments/sensors
  - Resolution, linearity, speed



- Digital-to-Analog

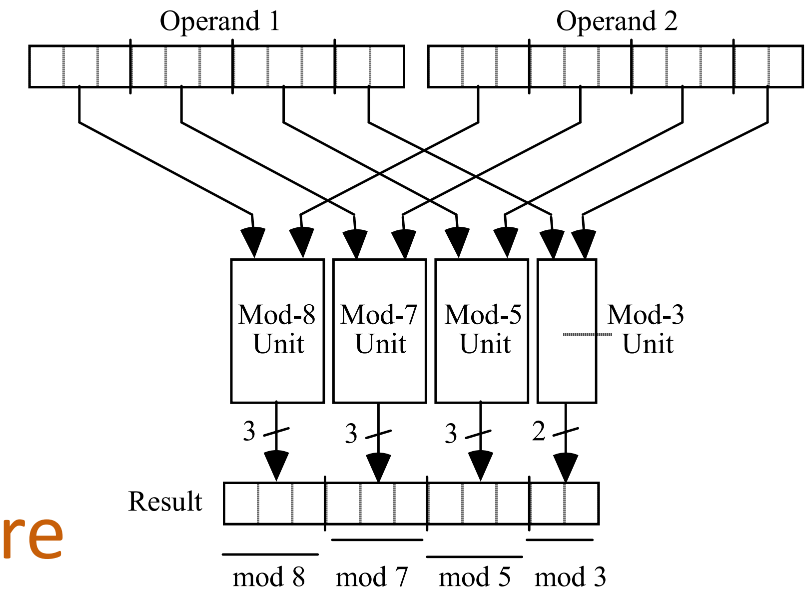


  - Sound output to speakers
  - Motor control (motion)
  - Resolution, range, speed



Residue Number System (RNS)

- Pairwise prime moduli: $m_{k-1} > \ldots > m_1 > m_0$
- Representation of x: $\{r_i = x \bmod m_i \mid 0 \le i \le k-1\}$
- RNS dynamic range: $M = \Pi_{0 \le i \le k-1} m_i$
  - Unsigned in $[0, M-1]$
  - Signed in $[-M/2, M/2-1]$
- RNS arithmetic algorithms
  - Digitwise add, sub, mult
  - Difficult div, sign test, compare



RNS: An Ancient Chinese Puzzle

Puzzle, due to the Chinese scholar Sun Tzu, 1500^{+} years ago:

What number has the remainders of 2, 3, and 2 when divided by 7, 5, and 3, respectively?

Residues (akin to digits in positional systems) uniquely identify the number, hence they constitute a representation: $(2 \mid 3 \mid 2)_{\text{RNS}(7|5|3)}$

In a weird way, RNS is a weighted representation

For RNS(7 | 5 | 3), the weights of the 3 positions are:

15 **21** **70**

Example -- Chinese puzzle: $(2 \mid 3 \mid 2)_{\text{RNS}(7|5|3)}$ represents the number

$$\langle \mathbf{15} \times 2 + \mathbf{21} \times 3 + \mathbf{70} \times 2 \rangle_{105} = \langle 233 \rangle_{105} = 23$$

RNS Representation

Puzzle, due to the Chinese scholar Sun Tzu, 1500^+ years ago:

What number has the remainders of 2, 3, and 2 when divided by 7, 5, and 3, respectively?

2 mod 7: 2 9 16 **23** 30 37 44 51

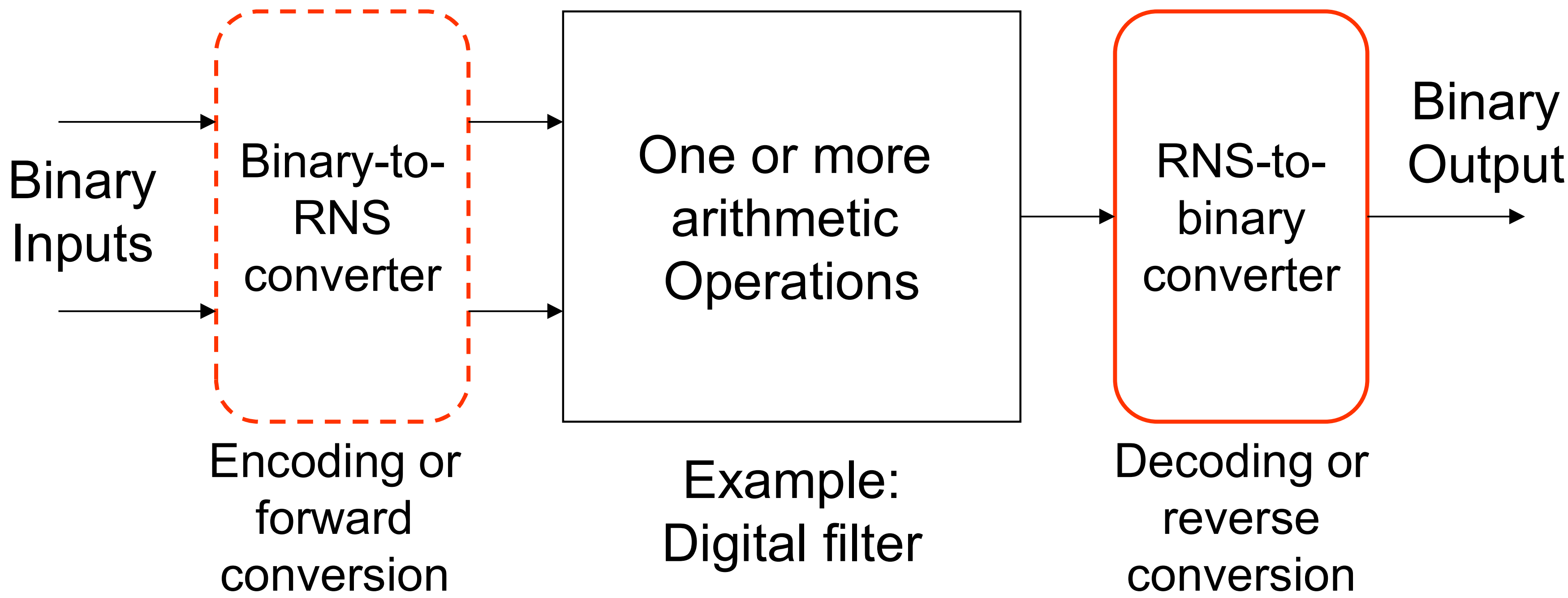
3 mod 5: 3 8 13 18 **23** 28 33 38 43 48

2 mod 3: 2 5 8 11 14 17 20 **23** 26 29 32 35 38 41 44 47 50

Each residue specifies a set of possible numbers or solutions to the puzzle

Intersection of the sets is the answer

Forward and Reverse Conversions



The more the amount of computation performed between the initial forward conversion and final reverse conversion (reconversion), the greater the benefits of RNS representation.

Three Related Abilities of the Brain

Sense of numbers

Basics wired-in by evolution

Advanced numeracy is learned

Sense of time

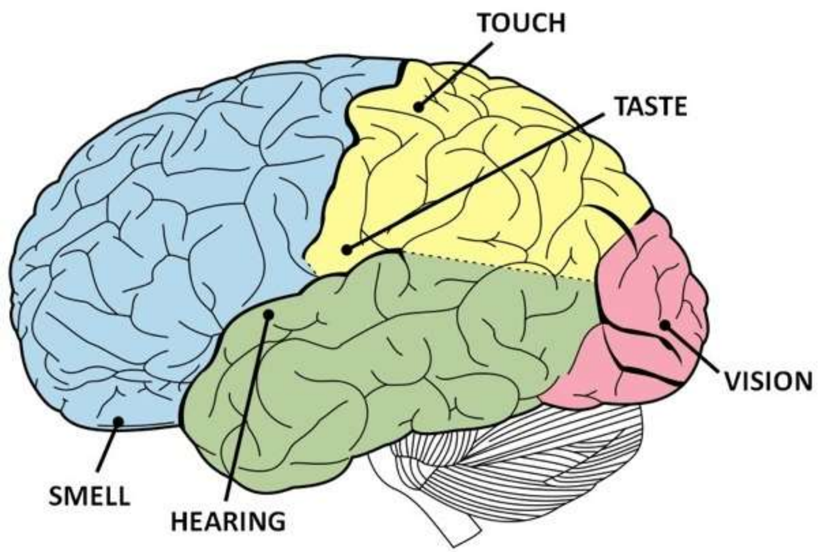
Your Brain Is a Time Machine

Mechanisms for different time scales

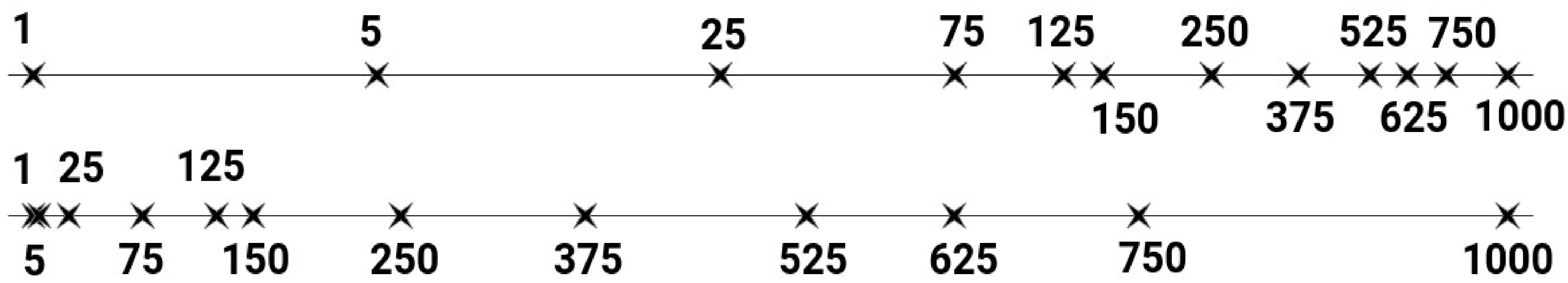
Sense of place

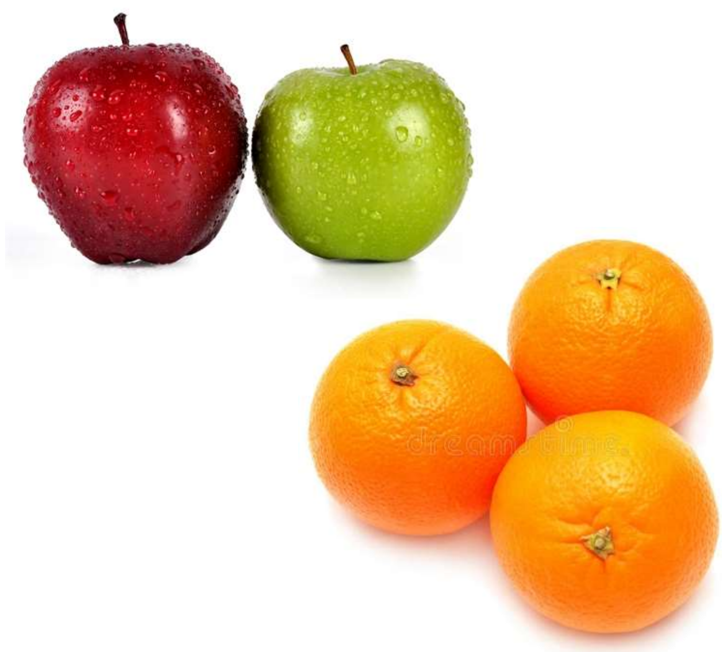
Location on different scales

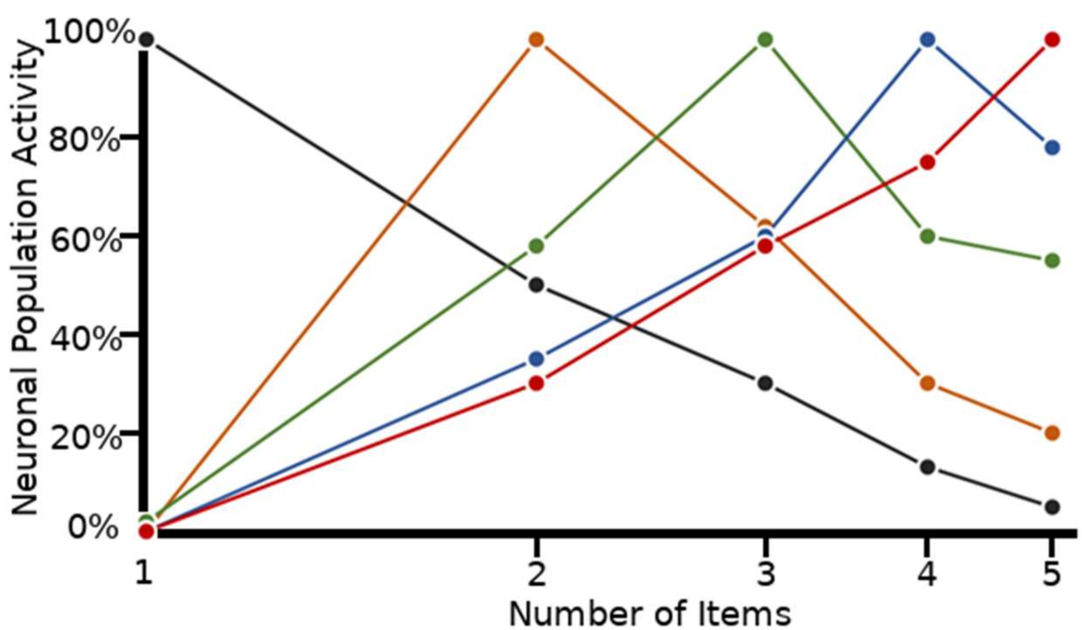
“Maps” of our surroundings



Sense of Numbers in Humans and Animals







Sense of Time in Humans and Animals

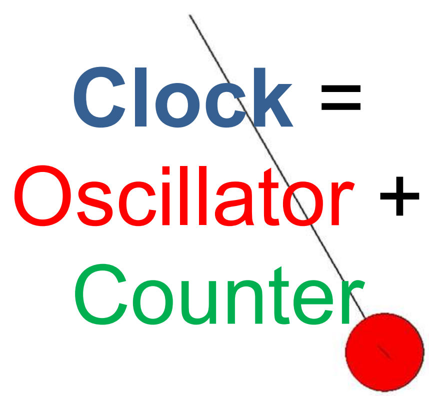
Current time, time duration, finishing on time

Remind me in 15 minutes; I’ll have to leave! (Set timer)

When did Amy arrive here? (Time duration, but in the past)

Circadian clock (day/night)

Periodic events (music)



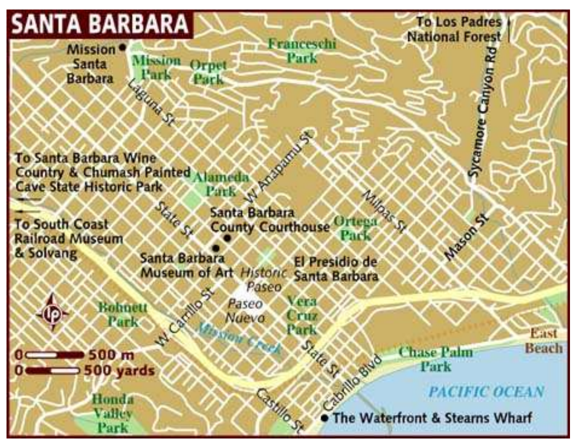
Following a schedule

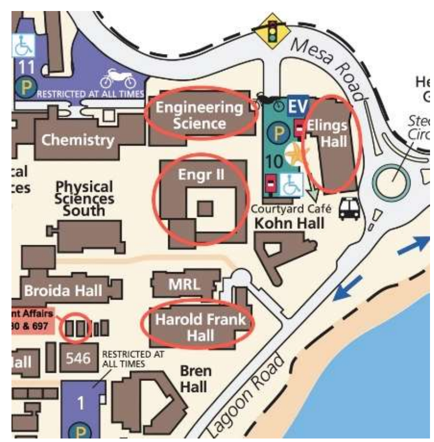
Sense of Place in Humans and Animals

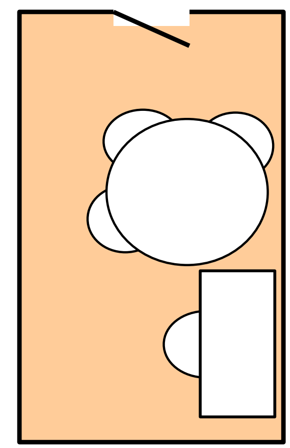
The sense of place and the ability to navigate are some of the most fundamental brain functions.

German philosopher Immanuel Kant (1724-1804) argued that some mental abilities exist independent of experience.

He considered perception of place as one of these innate abilities through which the external world had to be organized/perceived.







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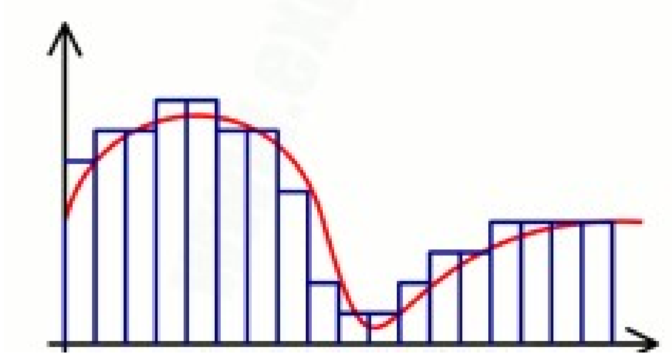
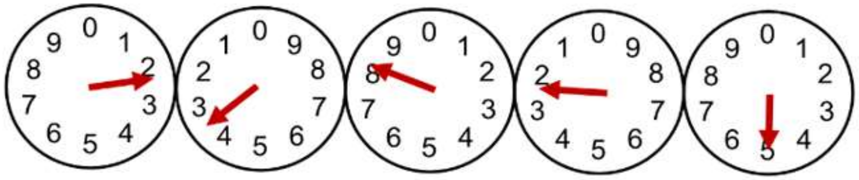
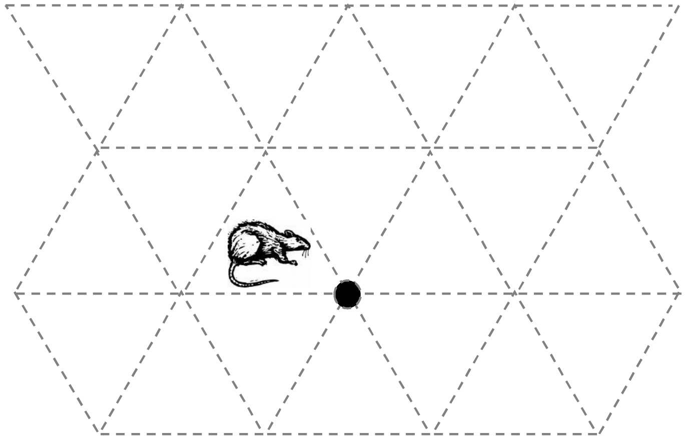






Quasi-Digital Parallel Counter

- Analog current summing
  - 7 inputs, 3-bit output
  - (*): Number of 1 inputs required to produce a 1
- The scheme is even older
  - Riordan and Morton, Use of Analog Techniques in Binary Arithmetic Units, *IEEE TC*, Feb. 1965

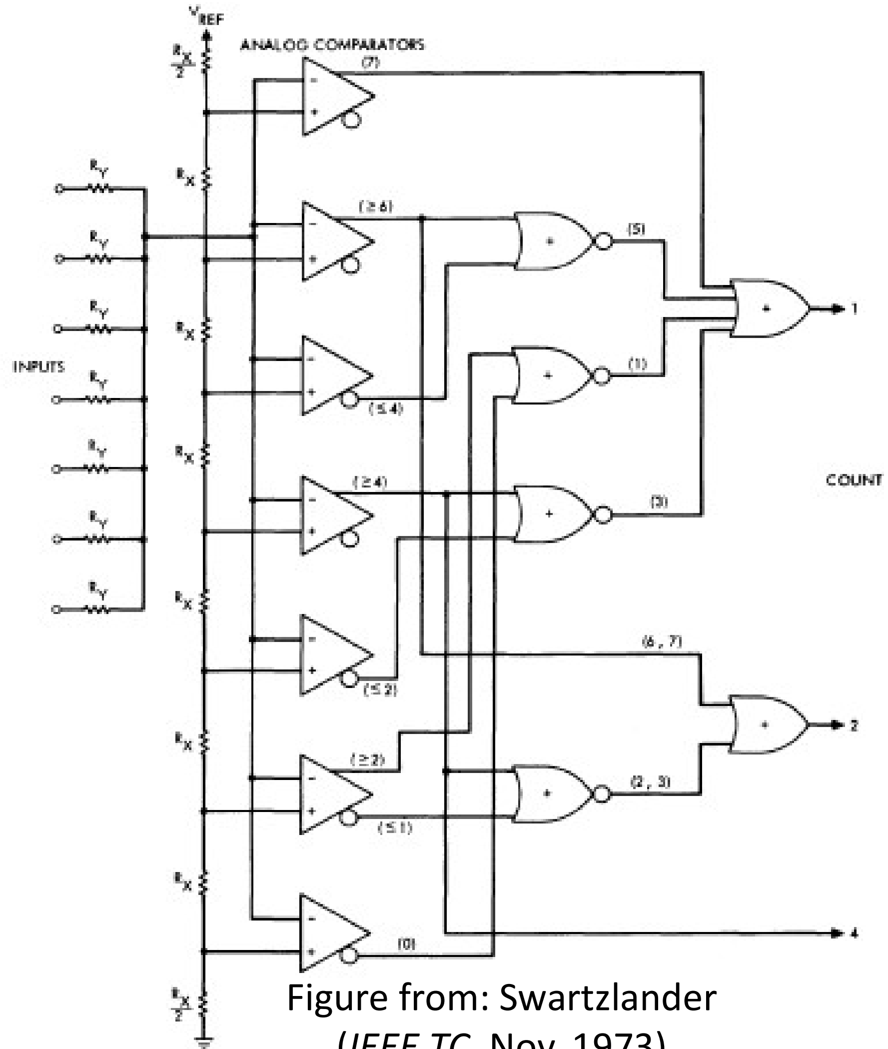
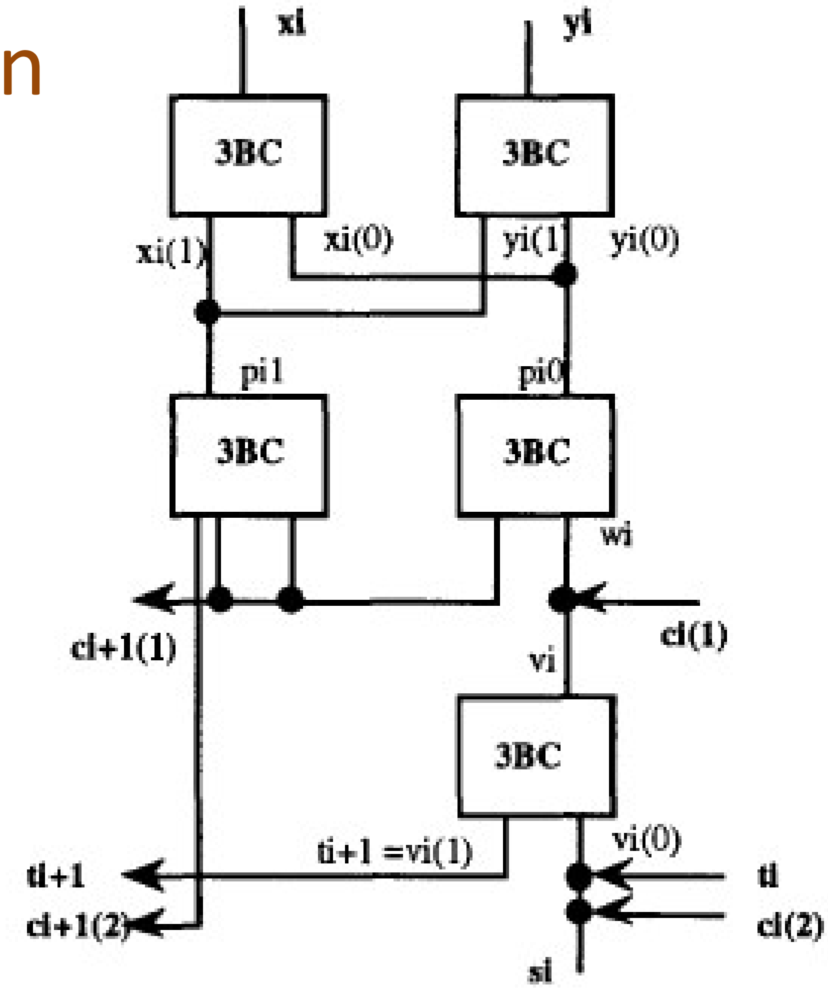


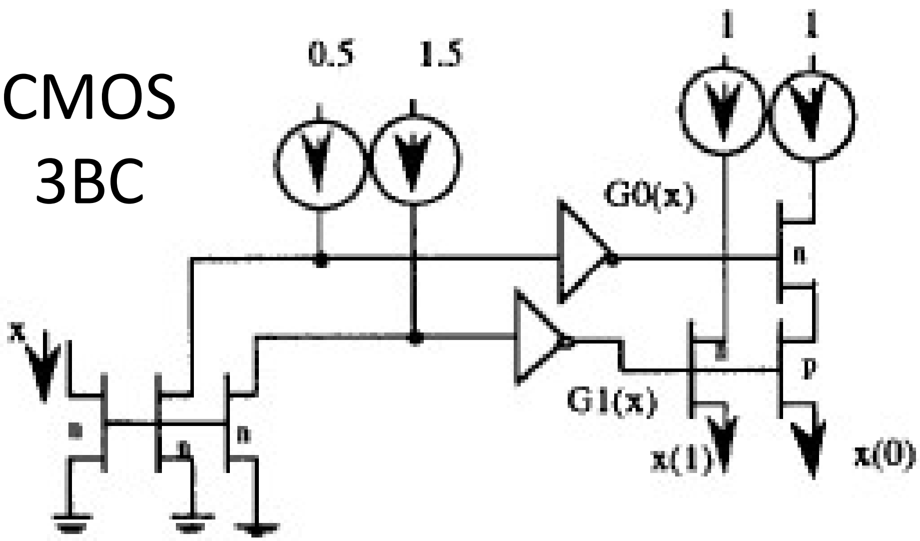
Figure from: Swartzlander (*IEEE TC*, Nov. 1973)

Current-Summing Multivalued Logic

- Binary stored-carry addition
  - Limited-carry algorithm
- 3-valued to binary conv.: 3BC

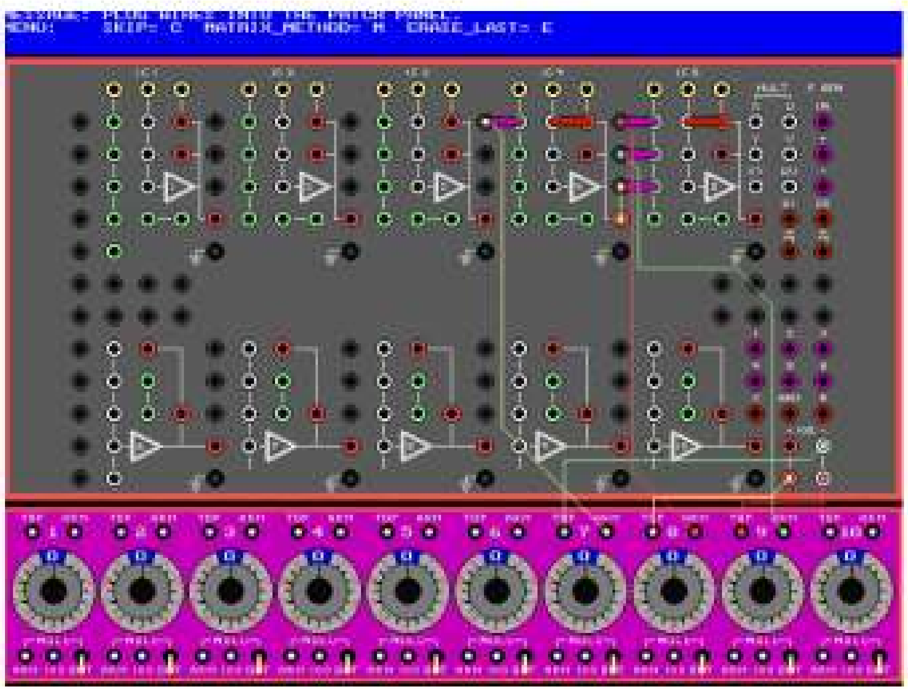


CMOS 3BC



Figures from: Etiemble & Navi (*SMVP*, May 1993)

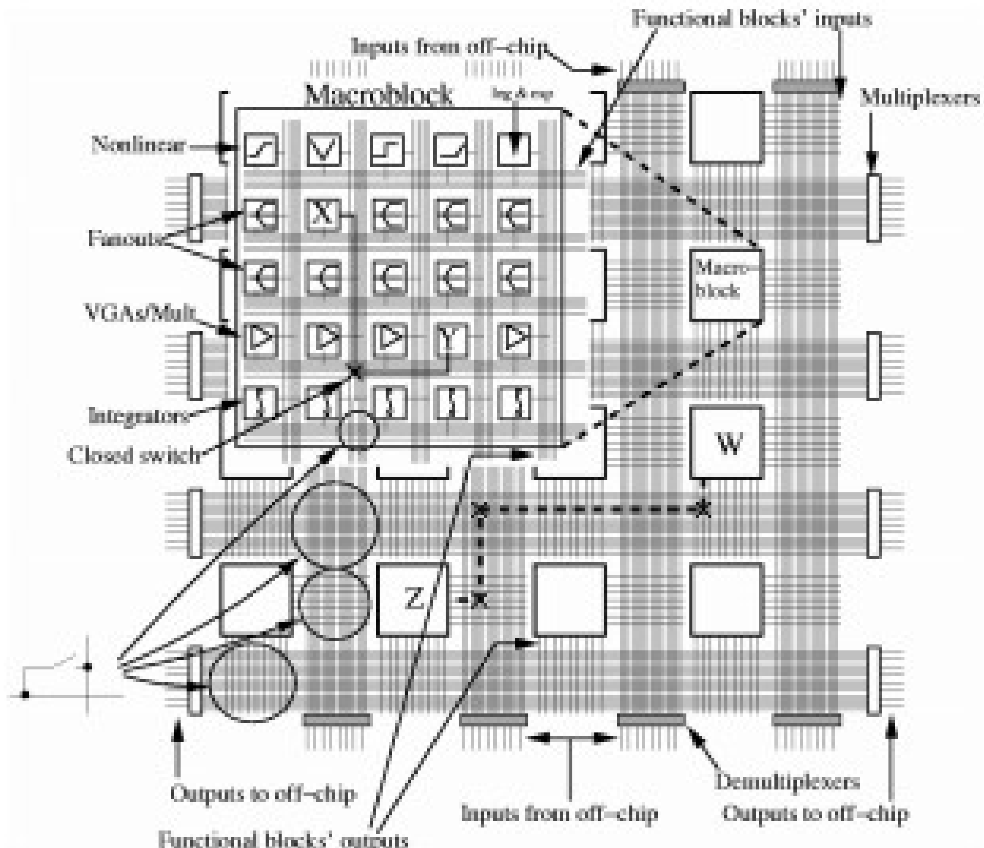
From Replacement to Resurgence



Patch panel of Systron-Donner Model 3400

Digital simulation of an analog computer

Analog co-processor unit for a digital CPU



Cowan/Melville/Tsividis, 2005

Hybrid Digital/Analog Representations

Continuous-digit residue number system (CD-RNS):
Inspired by how positional info encoding in rat's brain

Continuous-valued number system (CVNS):
A positional number system with analog digits

Race logic represents information as timing delays:
Results based on relative signal propagation delays

Space-time logic mimics spike-based neural nets:
Signal timing and magnitude both carry information

Mixed D/A Positional Representation

- Continuous-valued number system (CVNS)
  - Contains a form of natural redundancy
  - The MSD has all the magnitude info
  - Other digits provide successive refinements
- Familiar example: utility meter

3875 KWh

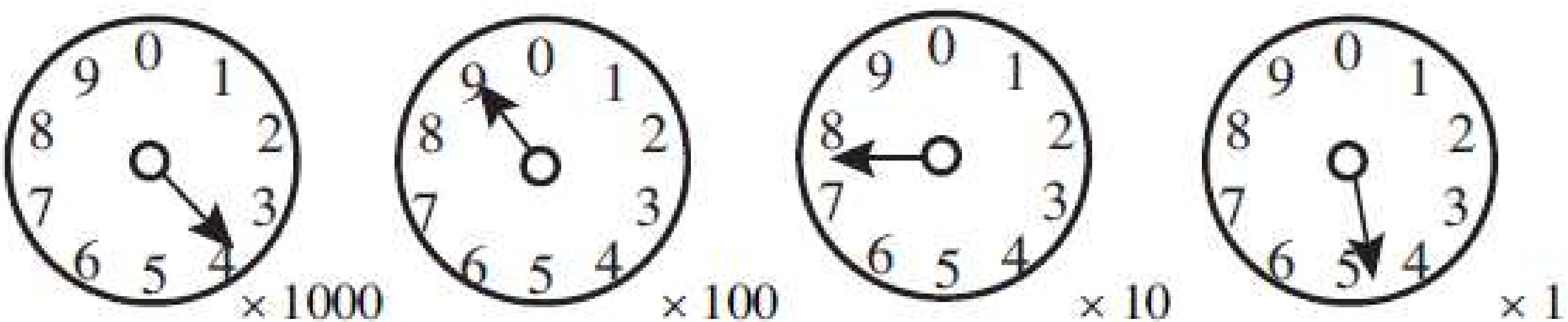


Figure from: Saed, Ahmadi, Jullien (*IEEE TC*, 2002)

Time-Delay-Based Race Logic

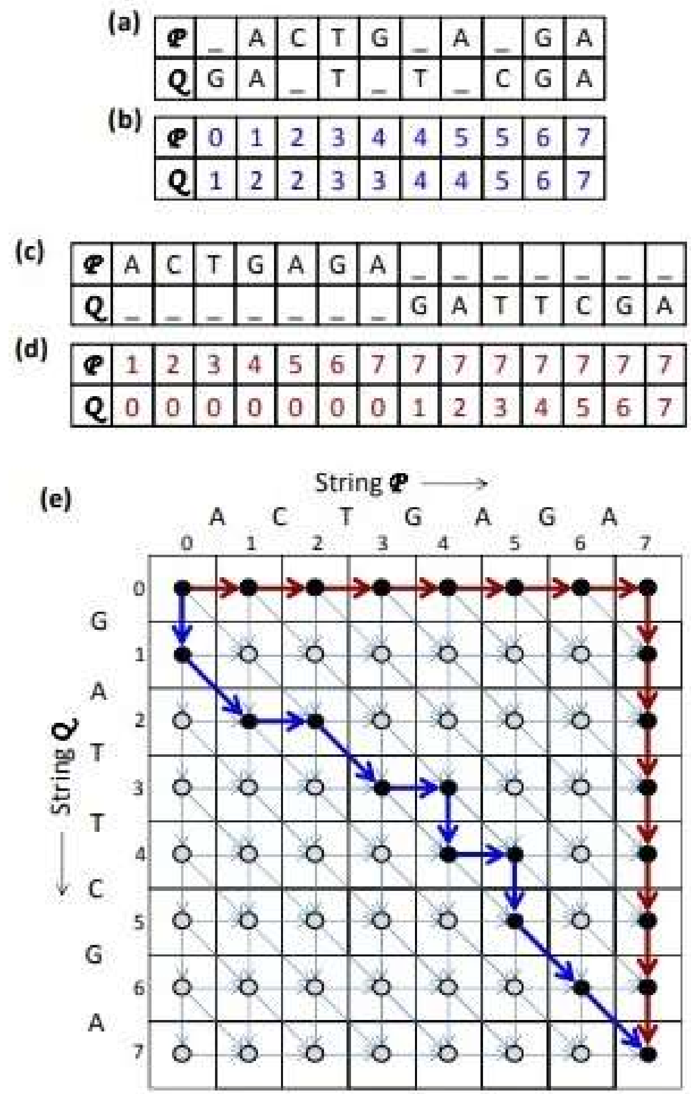
Not useful for general use (yet)

Quite efficient in some domains

Example: String alignment
(as in DNA/protein matching)
Closeness judged by “edit distance”

2D array of simple hardware cells
O(m^2) hardware complexity

Paths represent alignments
Horizontal, vertical, diagonal
moves have different latencies
Fastest path → Best alignment



Brain-Like Space-Time Computing

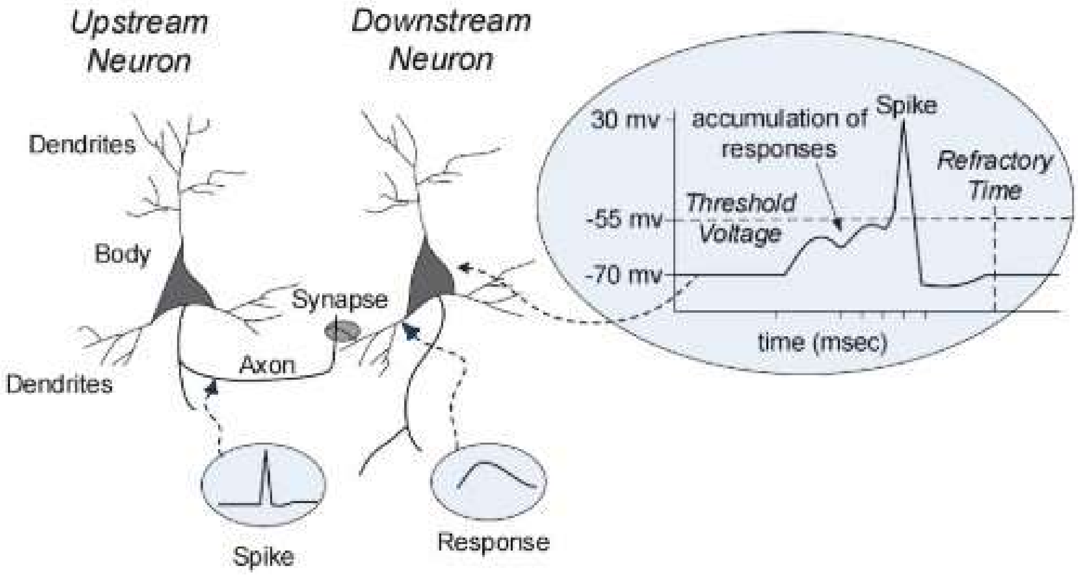
Example: Transmitting one 8-bit byte representing *n*

Binary: Delay = 9 slots; Average energy = 5 spikes

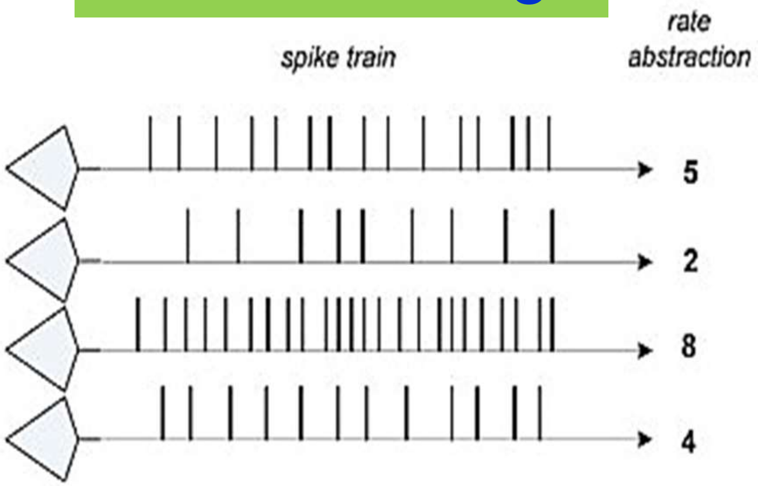
Start/synch spike, followed by 8 spikes/no-spikes

Space-time: Average delay = 130 slots; Energy = 2 spikes

Start/synch spike, followed by *n* no-spikes, then end spike

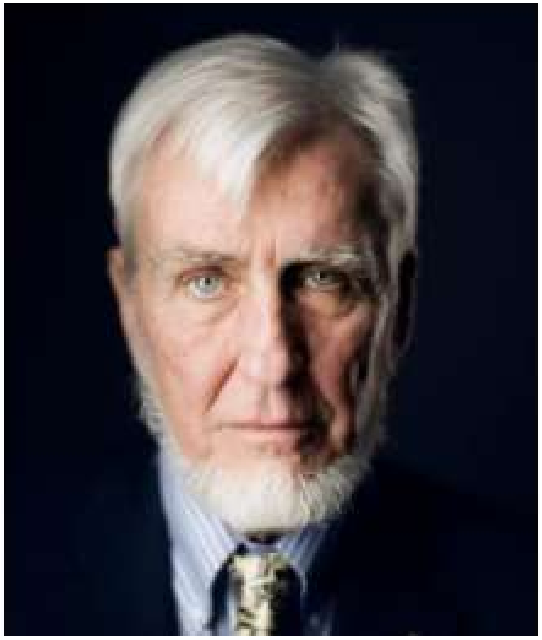


Rate encoding



Nobel Prize in Physiology or Medicine: 2014

One half went to John O'Keefe (University College, London), the other half to May-Britt Moser (Center for Neural Computation, Norway) and Edvard I. Moser (Kavli Institute for Systems Neuroscience, Norway) "for their discoveries of cells that constitute a positioning system in the brain."

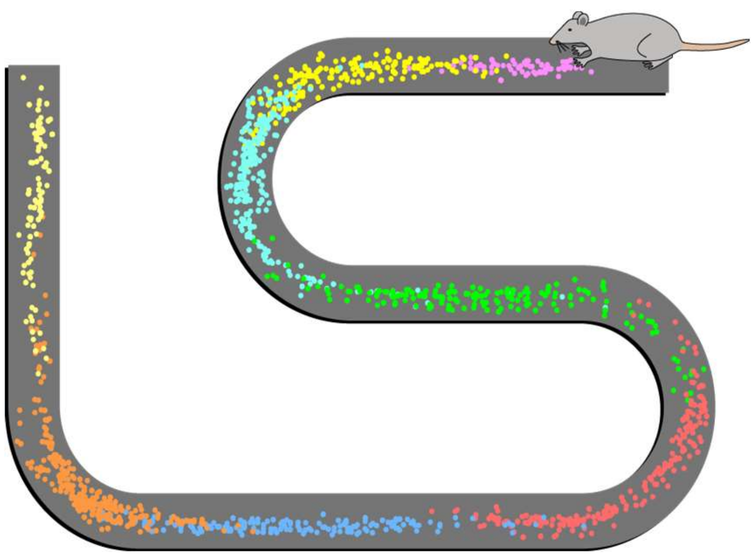


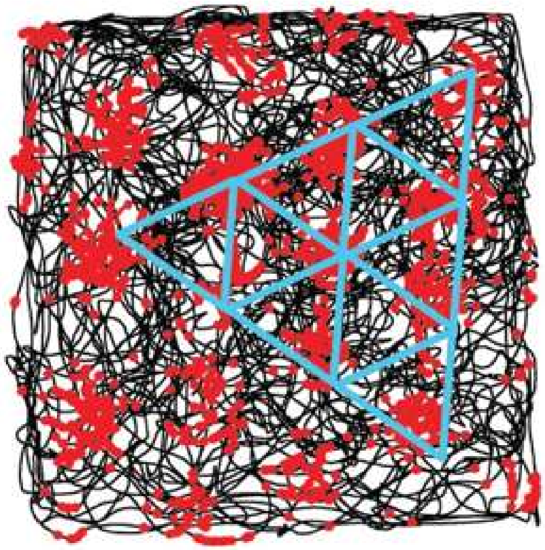
The Nobel Laureates' Contributions

John O'Keefe discovered place cells in the hippocampus that signal position and provide the brain with spatial memory capacity.

May-Britt Moser and Edvard I. Moser discovered in the medial entorhinal cortex, a region of the brain next to hippocampus, grid cells that provide the brain with a coordinate system for navigation.

Place cells firings (image from Wikipedia)





Grid cells firings (image from Moser/Rowland/ Moser, 2015)

First Attempt at Understanding

6858 • The Journal of Neuroscience, July 2, 2008 • 28(27):6858 – 6871

Behavioral/Systems/Cognitive

What Grid Cells Convey about Rat Location

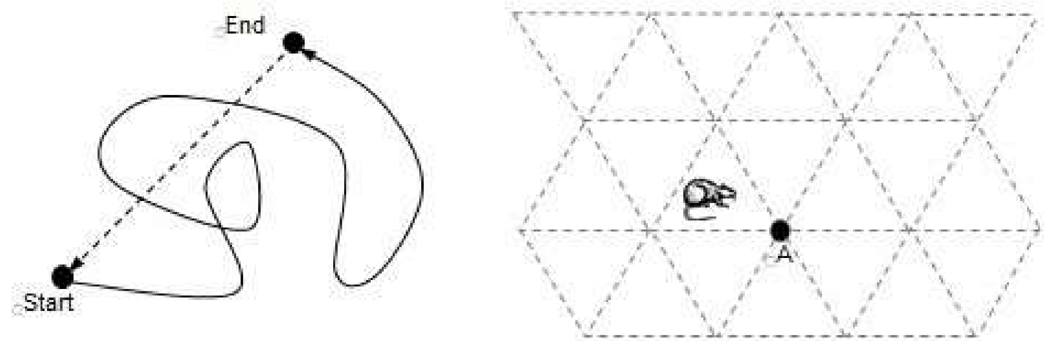
Ila R. Fiete,[1,3] Yoram Burak,[1,4] and Ted Brookings[2,5]

[1]Kavli Institute for Theoretical Physics and [2]Department of Physics, University of California, Santa Barbara, Santa Barbara, California 93106, [3]Broad Fellows Program, Computation and Neural Systems, California Institute of Technology, Pasadena, California 91125, [4]Swartz Fellows Program, Center for Brain Science, Harvard University, Cambridge, Massachusetts 02138, and [5]Volen Center, Department of Biology, Brandeis University, Waltham, Massachusetts 02454

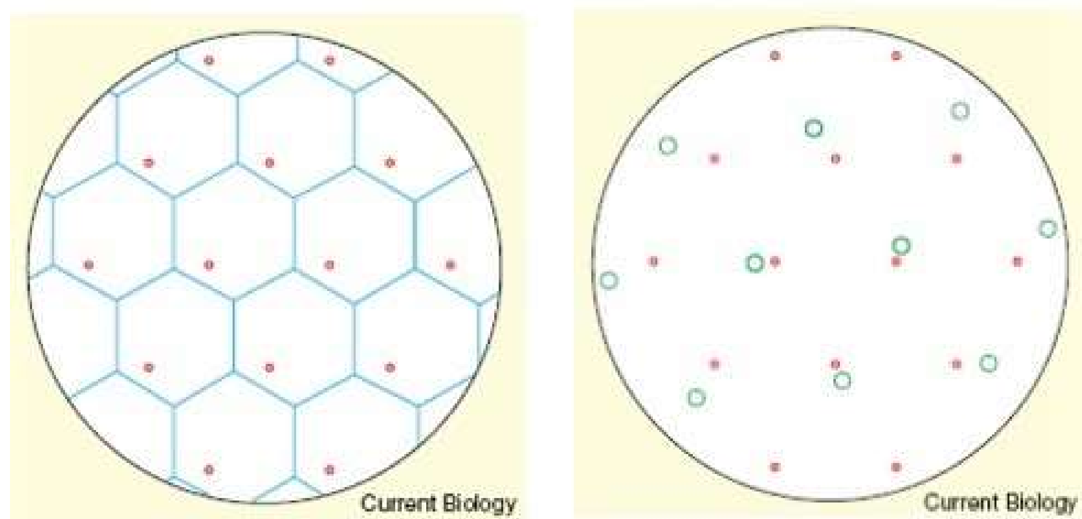
We characterize the relationship between the simultaneously recorded quantities of rodent grid cell firing and the position of the rat. The formalization reveals various properties of grid cell activity when considered as a neural code for representing and updating estimates of the rat's location. We show that, although the spatially periodic response of grid cells appears wasteful, the code is fully combinatorial in capacity. The resulting range for unambiguous position representation is vastly greater than the ≈1–10 m periods of individual lattices, allowing for unique high-resolution position specification over the behavioral foraging ranges of rats, with excess capacity that could be used for error correction. Next, we show that the merits of the grid cell code for position representation extend well beyond capacity and include arithmetic properties that facilitate position updating. We conclude by considering the numerous implications, for downstream readouts and experimental tests, of the properties of the grid cell code.

Localization with Grid Cells

- Rat’s navigation system
  - Wavy travel path
  - Straight return path
  - Even in the dark
- Nervous system has place cells & grid cells
  - Grid cell firings
  - Relative in-cell position
- In-cell positions within several grids pinpoints rat’s absolute location



(a) Travel and return paths (b) Rat’s hexagonal grid



(c) Firings and locations [3] (d) Two hexagonal grids [3]

The Questions to Be Addressed

- A rat can go up to a certain distance and still be able to find its way back (range)
  - Translating grid-cell firings to spatial information
  - How the range is related to grid-cell parameters
  - Representation range vs. the observed distance
- Fiete, Burak, and Brookings had connected the grid cells to residue representation
  - Couldn't confirm the hypothesis theoretically
  - Relied on extensive simulation for confirmation

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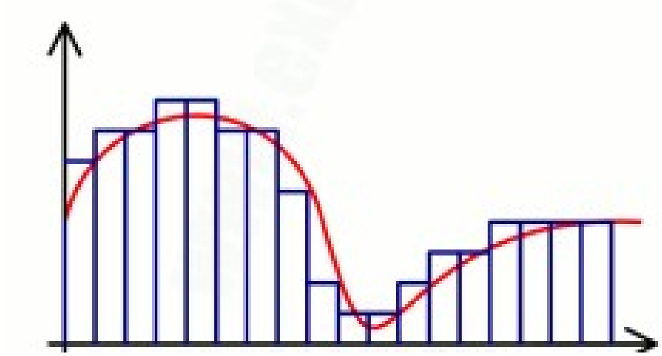

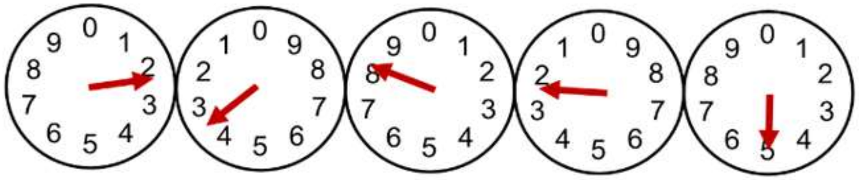

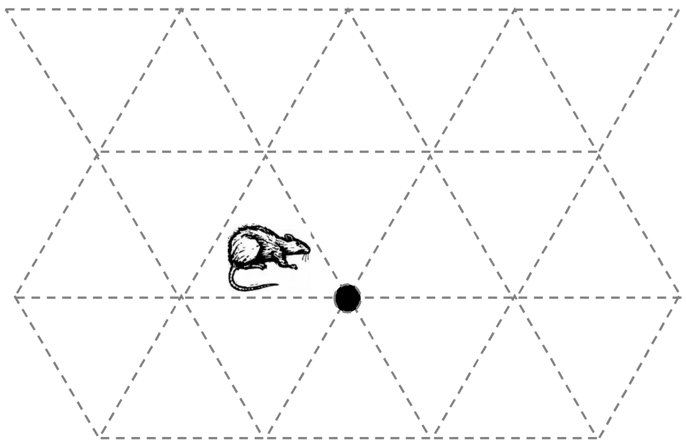

My First Contribution to the Problem

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Digital Arithmetic in Nature: Continuous-Digit RNS

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It has been reported in the literature on computational neuroscience that a rat's uncanny ability to dash back to a home position in the absence of any visual clues (or in total darkness, for that matter) may stem from its distinctive method of position representation. More specifically, it is hypothesized that the rat uses a multimodular method akin to residue number system (RNS), but with continuous residues or digits, to encode position information. After a brief review of the evidence in support of this hypothesis, and how it relates to RNS, we discuss the properties of continuous-digit RNS, and derive results on the dynamic range, representational accuracy and factors affecting the choice of the moduli, which are themselves real numbers. We conclude with suggestions for further research on important open problems concerning the process of selection, or evolutionary refinement, of the set of moduli in such a representation.

RNS with Analog Digits (Remainders)

- I formulated the spatial representation problem with the grid cells as CD-RNS
  - First time RNS is used with analog remainders
  - Conventional RNS theory is inapplicable
  - I developed a theory for CD-RNS and its range
- Analog and mixed digital-analog technology has a long history in computer arithmetic
  - Digital so good that these were not pursued
  - More use of analog features expected to come

CD-RNS Representation

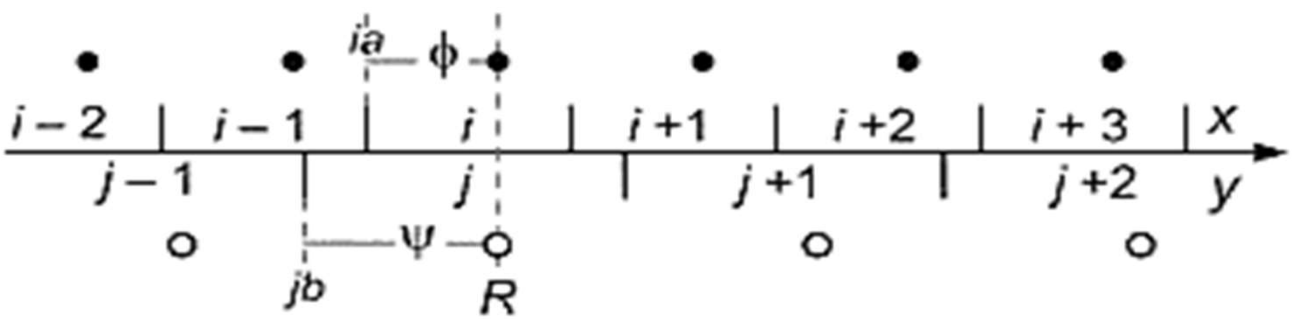
Modular arithmetic with continuous residues

Extension of RNS to non-integer moduli and residues

- Offers precision-range-robustness trade-offs
- More accurate residues widen the range and increase robustness

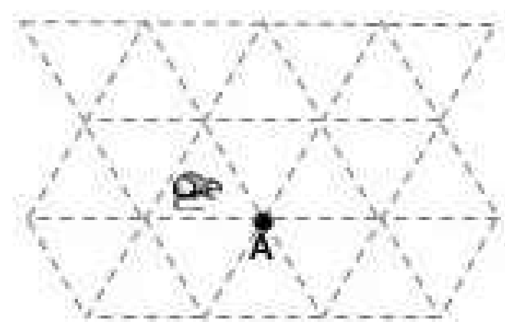
Of interest to neuroscientists: Rat's navigation system

- Rat uses "space cells" (absolute) and "grid cells" (relative)
- Can return to home position in the dark, without any visual cues



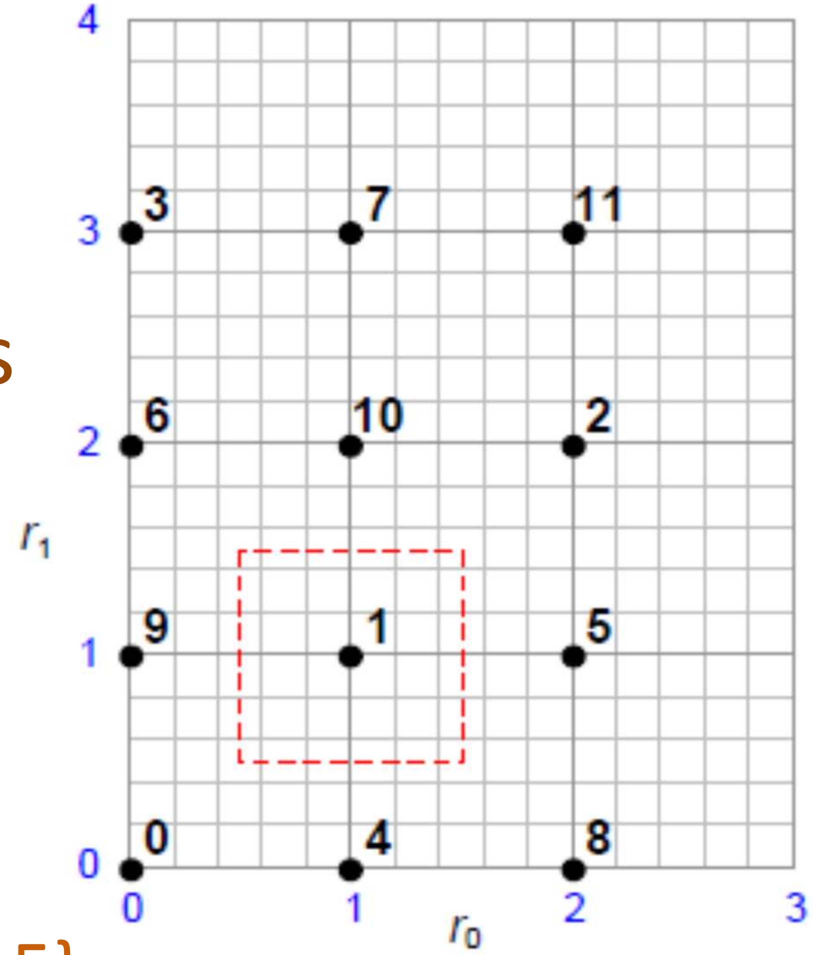
Localization with 2 grids in 1D space

Rat's hex grid



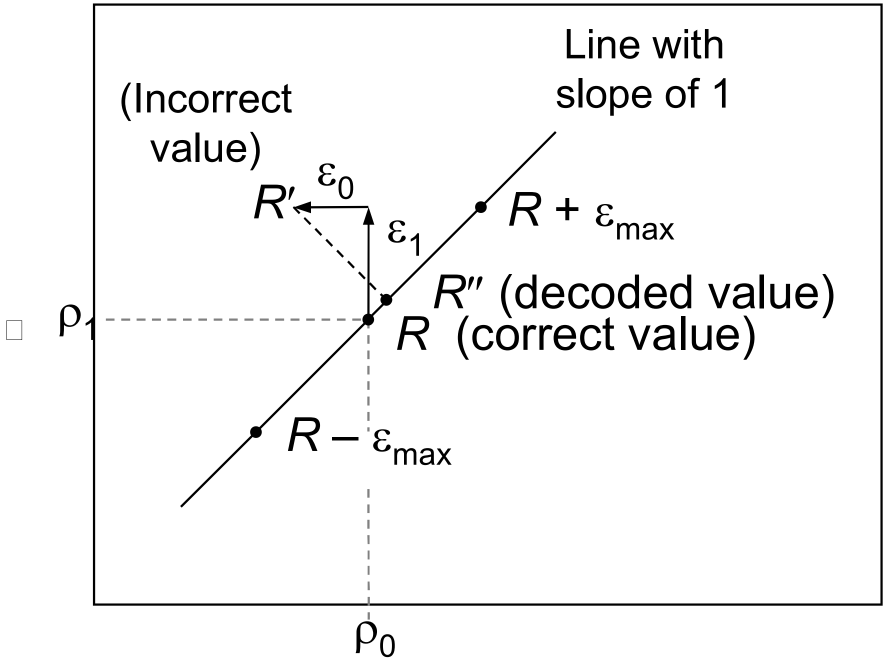
Integer Moduli and Residues

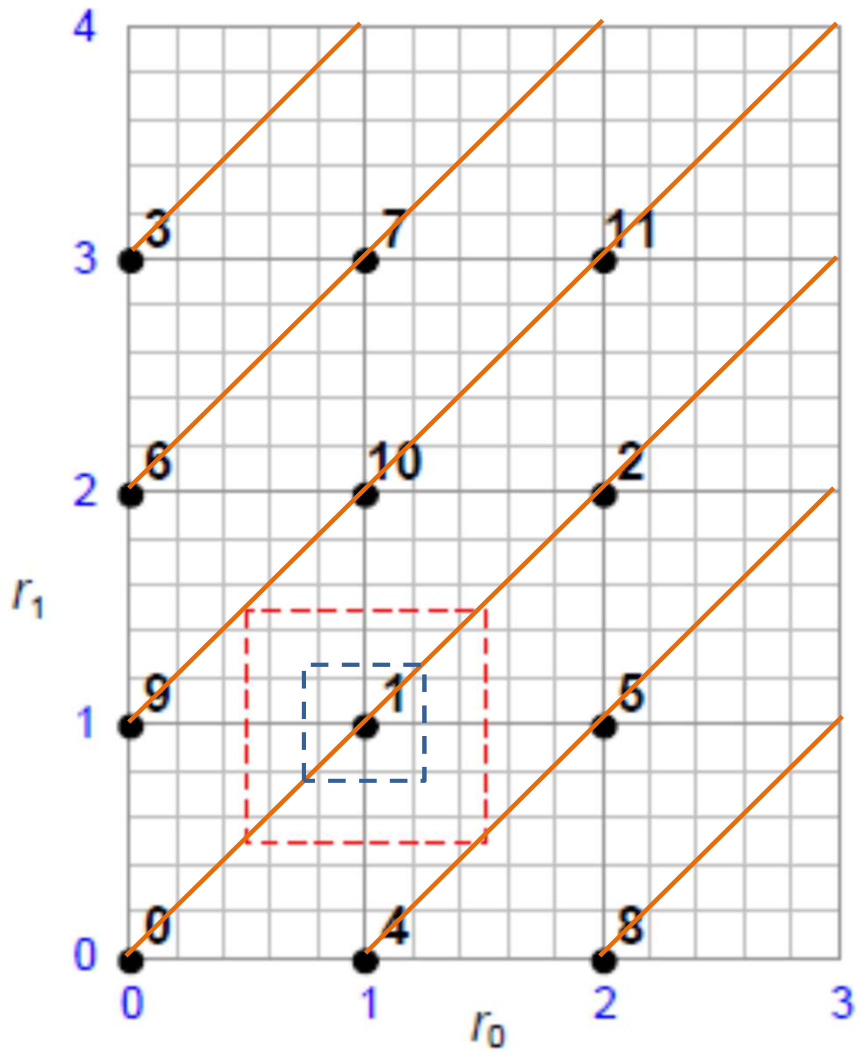
- Two-modulus RNS {4, 3}
- Dynamic range [0, 11]
- Imagine residues with errors
  - Errors < 0.5 correctable
  - Errors < 1.0 detectable
- Multiresidue systems
  - 3-modulus RNS {5, 4, 3}
  - {5,4,3} ≡ {20,3} ≡ {15,4} ≡ {12,5}



Integer Moduli, Continuous Residues

- Residue errors ε_1 and ε_0
- Decoding error $\leq \max(\varepsilon_1, \varepsilon_0)$
- Dynamic range? $[0, 12 - \varepsilon_{max}]$
- Max allowable error < 0.25





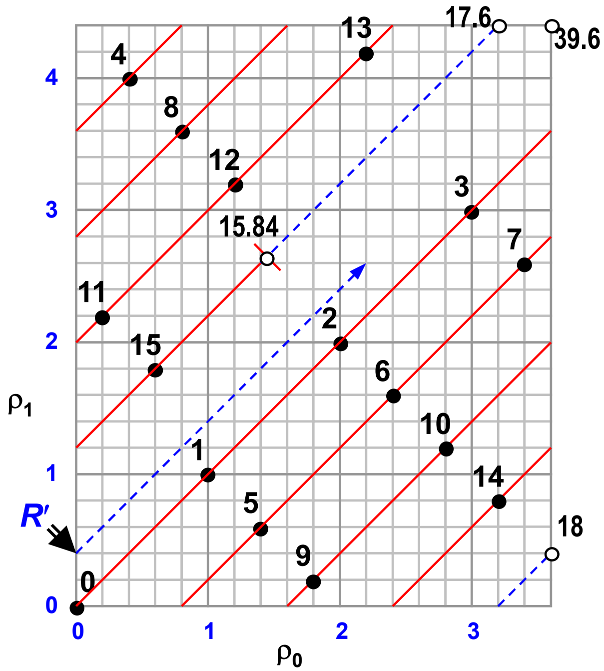
Range-Precision Trade-off in CD-RNS

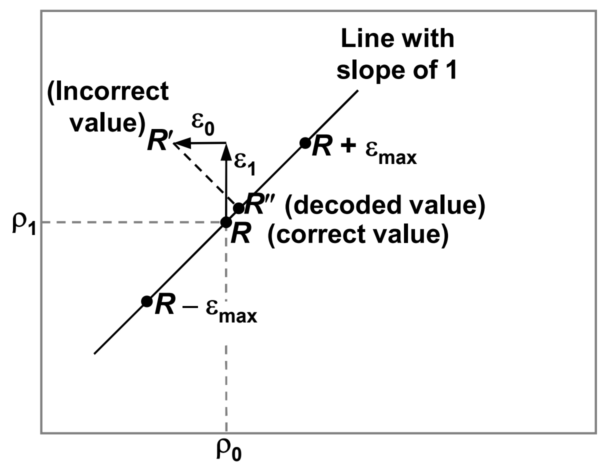
Dynamic range is proportional to the product of moduli, divided by maximum error

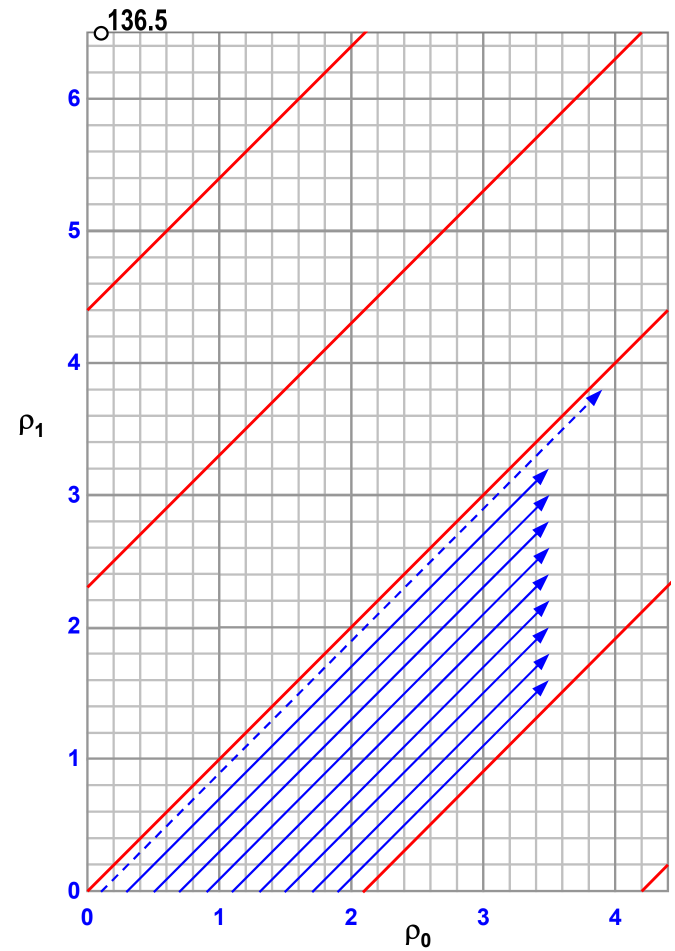
Left diagram: Range extension beyond Πm_i

Middle diagram: Decoding error

Right diagram: CD-RNS with $m_1 = 6.5$, $m_0 = 4.4$







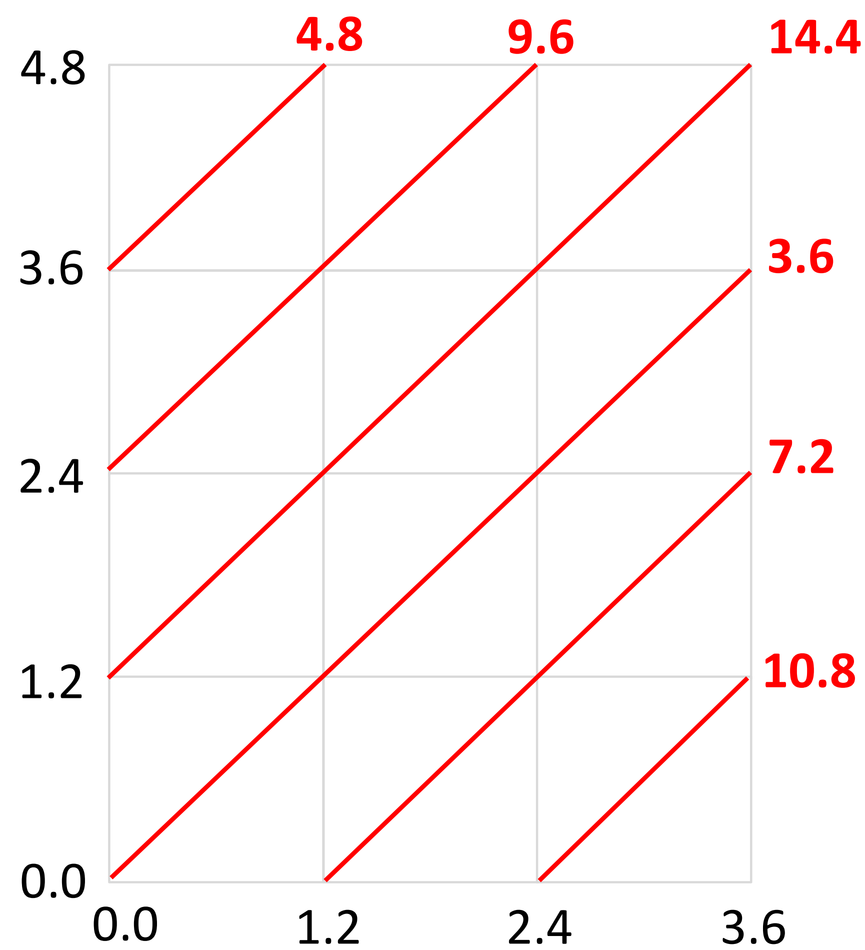
Continuous Moduli and Residues

Case 1: The moduli are integer multiples of their difference

With proper scaling, the CD-RNS can be converted to an RNS

This example is equivalent to RNS {4, 3} with scale factor 1.2

Question:
Are there CD-RNSs that cannot be replaced with ordinary RNSs?

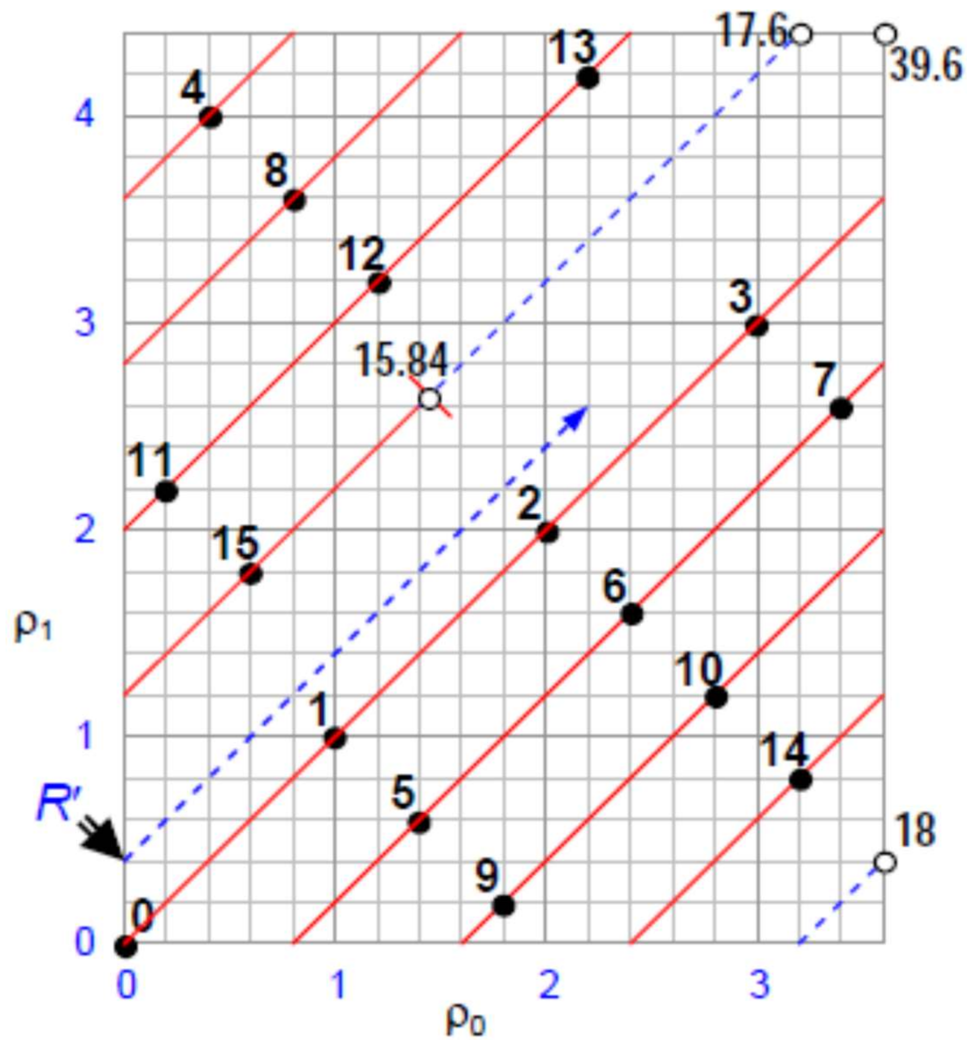


Equivalence of CD-RNS and RNS

Case 2a: The moduli are integer multiples of some number *s* (that divides their difference)

With proper scaling, the CD-RNS can be converted to an RNS, provided max error target is ≤ *s*/4

For this example, *s* = 0.4 and the system is equivalent to RNS {11, 9} with scale factor 0.4

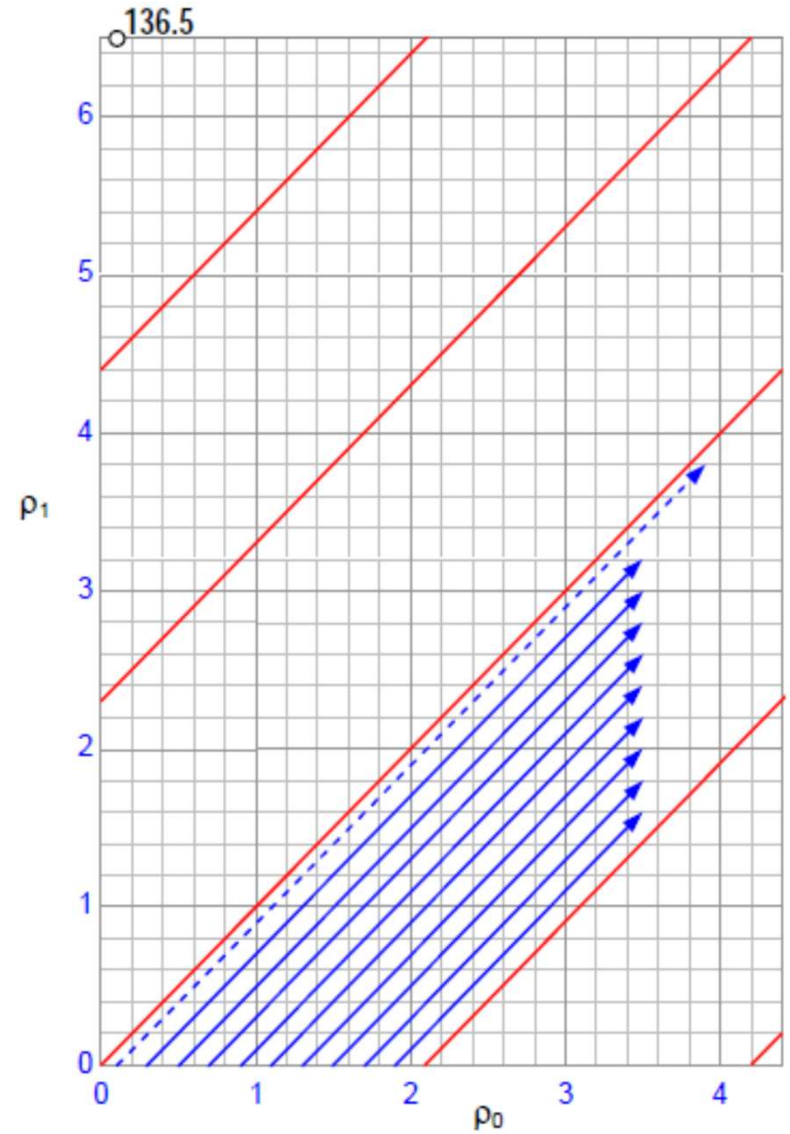


Representational Power of CD-RNS

Case 2b: The moduli are integer multiples of some number *s* (that divides their difference), but max error target > *s*/4

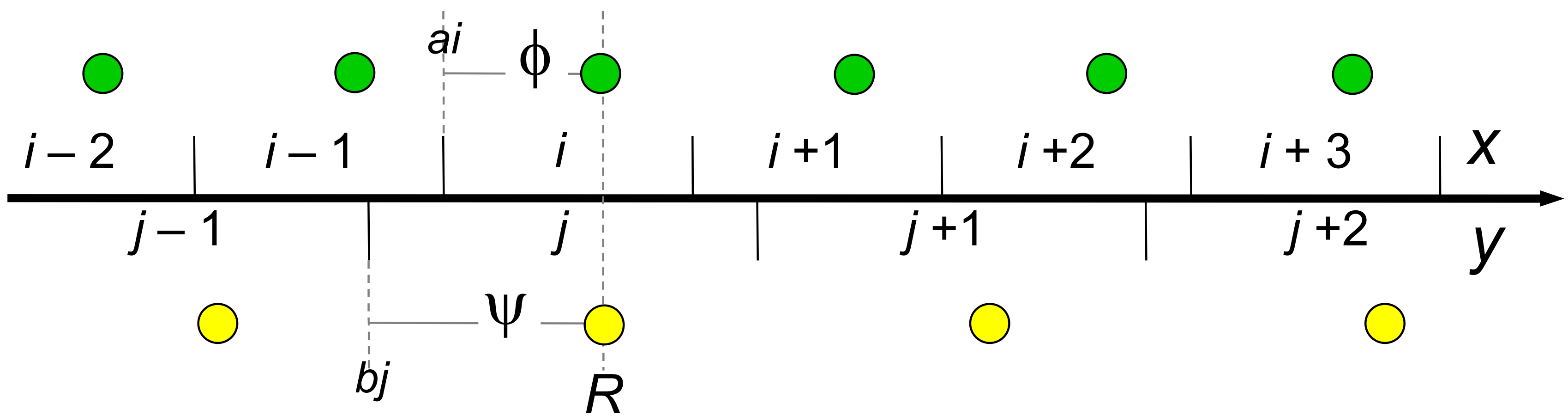
The CD-RNS is not equivalent to an RNS in terms of representational capability and dynamic range

For this example, *s* = 0.1 but the system is different from RNS {65, 44} with scale factor 0.1



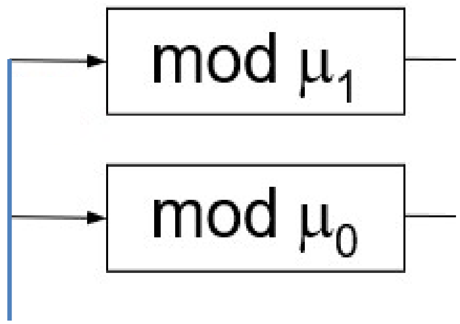
Conceptually Simpler 1D Example

- Distance encoded by mod-*a* and mod-*b* residues
  - Phases ϕ and ψ given
  - Reverse conversion provides *R*
- *R* is a point whose mod-*a* and mod-*b* residues match ϕ and ψ to within the error bound



Forward Conversion to CD-RNS

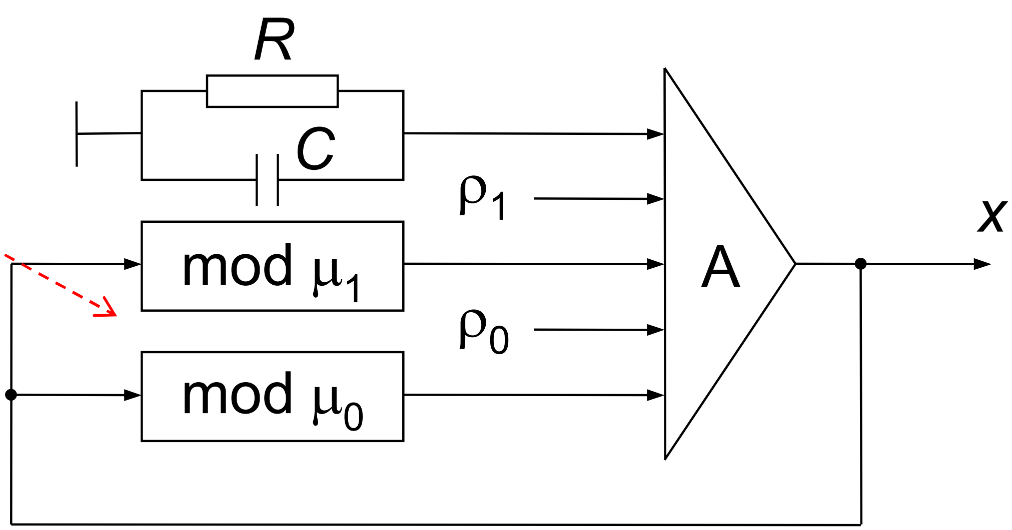
- Quite similar to ordinary RNS
  - What multiple of the modulus should be subtracted?
  - Multiple comparators (like those in A/D converters)
  - Use binary search to reduce hardware complexity
- Example: mod-4.4, with dynamic range 35.0
  - Compare with 17.6; Subtract if greater
  - Compare with 8.8; Subtract if greater
  - Compare with 4.4; Subtract if greater
  - Output the remainder



Backward Conversion to Binary

- CRT and its derivatives are inapplicable
  - Conversion amplifies the errors
  - Example 15 in my 2015 *Computer Journal* paper
- View the conversion as nonlinear optimization
  - Convergence occurs with circuit's RC time constant

Forward conversion

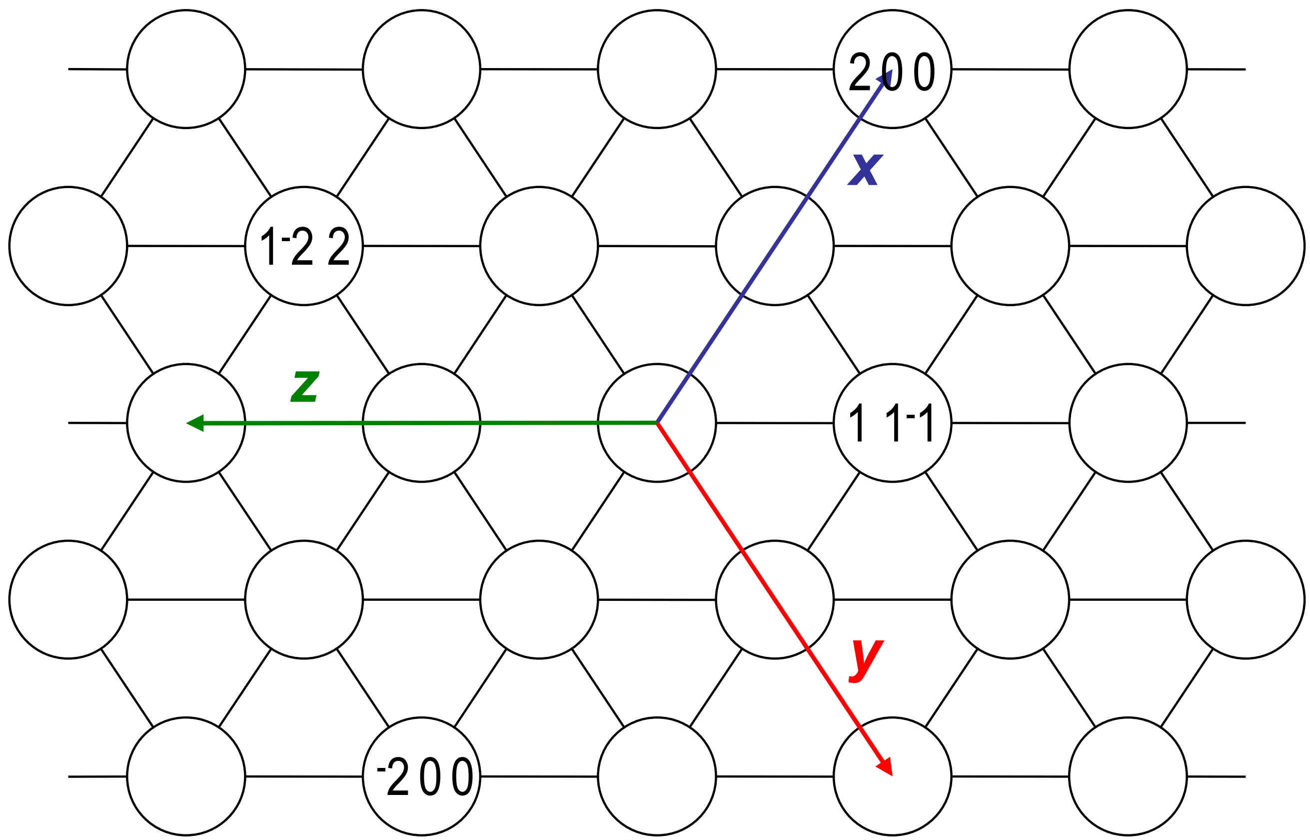


Sun & Yao, *IEEE Int'l Conf. Neural Networks*, 1994

Hex Grid Coordinate System

- Point identified by 3 coordinates, one of which is redundant

- Redundancy allows error correction beyond the system's accuracy range



Open Problems in Neurobiology

- Dynamic range of rat's navigation system
- Numerical simulation: Range ~ $(1/\varepsilon_{max})^{Exponent}$

  Exponent ≈ Number of moduli – θ
- **Example:** 12 moduli ⇒ Exponent = 10.7

  Our results yield an exponent of 11.0
- How did the rat's navigational grids evolve?

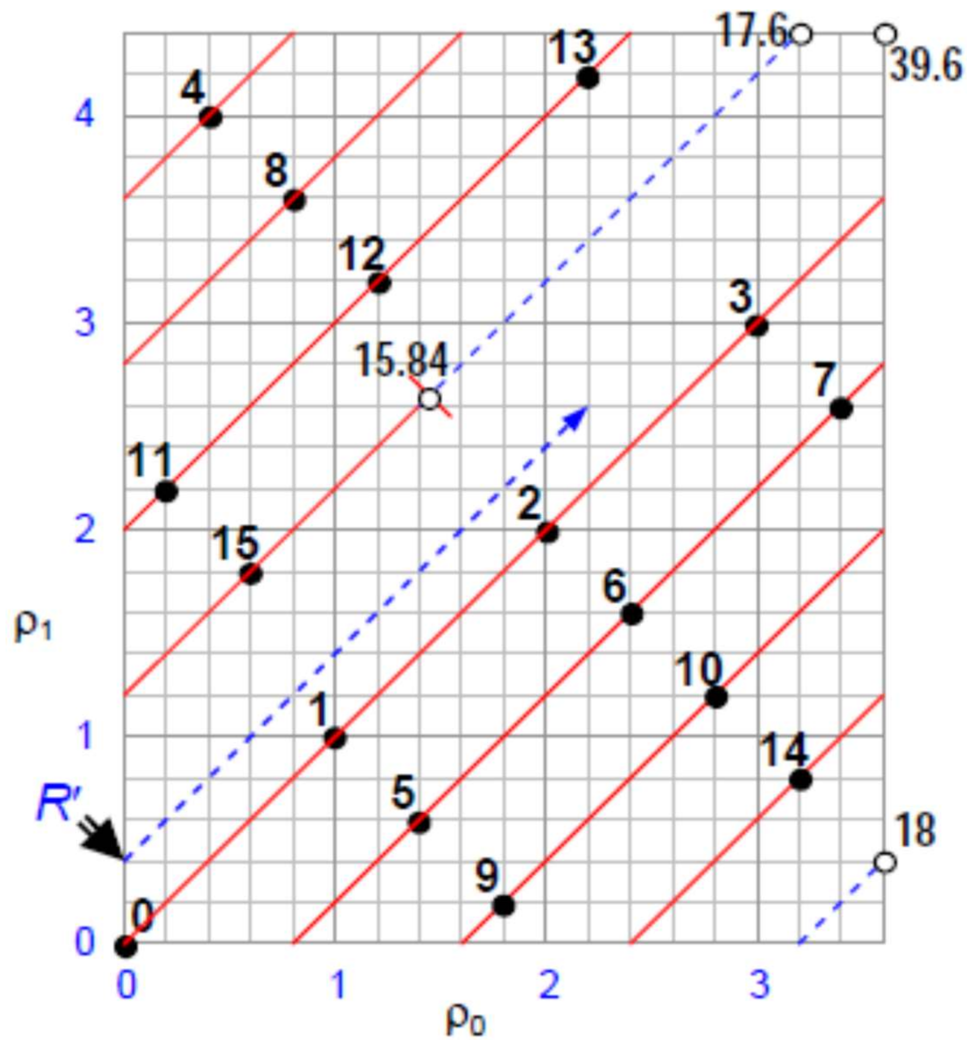
  (Evolutionary basis for moduli optimization)

Choosing the CD-RNS Moduli

Theorem 2: $\mu \geq 36.0$
Theorem 3: $\mu \leq 39.6$

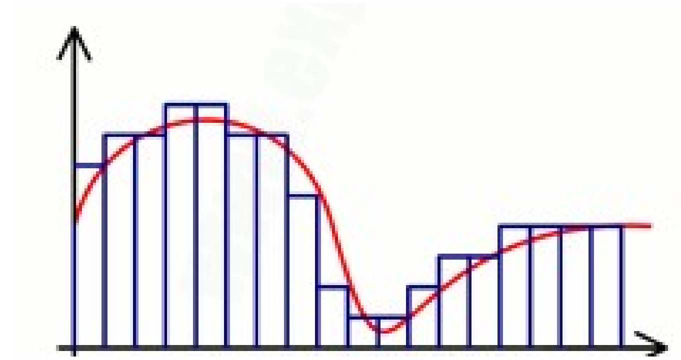
Intuitively, the moduli are optimal when the two bounds coincide

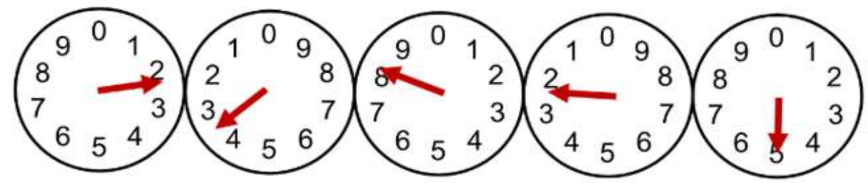
To cover the dynamic range μ, choose the moduli that are on the order of $(2\mu\varepsilon_{max})^{1/2}$ and differ by $2\varepsilon_{max}$

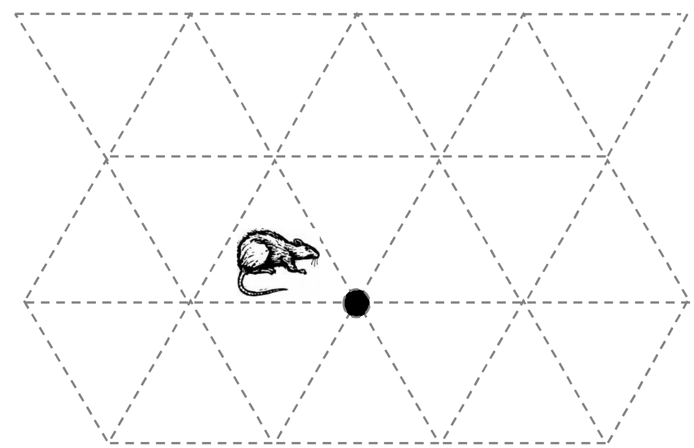


Outline

- **Introduction and Background**
  - Why analog is cool again
  - Residue number system (RNS)
  - The brain and numeracy
- **Hybrid Numbers and Arithmetic**
  - Historical perspective
  - Some hybrid representations
  - Nobel Prize in medicine, 2014
- **Continuous-Digits RNS**
  - CD-RNS models rat’s navigation
  - Different from discrete RNS
  - Dynamic range and precision
  - Other properties and challenges
- **Conclusions and Future Work**







Summary and Conclusions

- Introduced RNS with continuous residues
  - Distinct from ordinary RNS
  - Advantages (similar to other hybrid schemes)

- Studied range, accuracy, tradeoffs
  - Tight bounds for dynamic range
  - Optimal choice of moduli

- Showed link to computational neuroscience
  - Rat's sense of location, navigation
  - Moduli in nature: evolutionary implications

Ongoing and Future Work

- Refine and extend the theoretical framework
  - Arithmetic and algorithmic implications
  - Exact dynamic range, or even tighter bounds
- Study development and application aspects
  - Circuit realization & building blocks
  - Latency, area, energy implications

- Pursue links with other D/A methods
  - Mixed implementations?

Wrapping Up: The Big Picture

Analog computing is making a comeback and hybrid digital-analog computing is becoming more attractive

D-A computing can be combined at various levels:
- Representation level, as in CVNS and CD-RNS
- Analog approximation, digital refinement
- Neuromorphic computing paradigm
- Multi-level combination methods

D-A

Future work and more detailed comparisons
- Assessment of relative speeds in application contexts
- Quantifying cost and energy requirements
- Effects of radix and moduli selection
- Other D/A combination methods

Thank You for Your Attention

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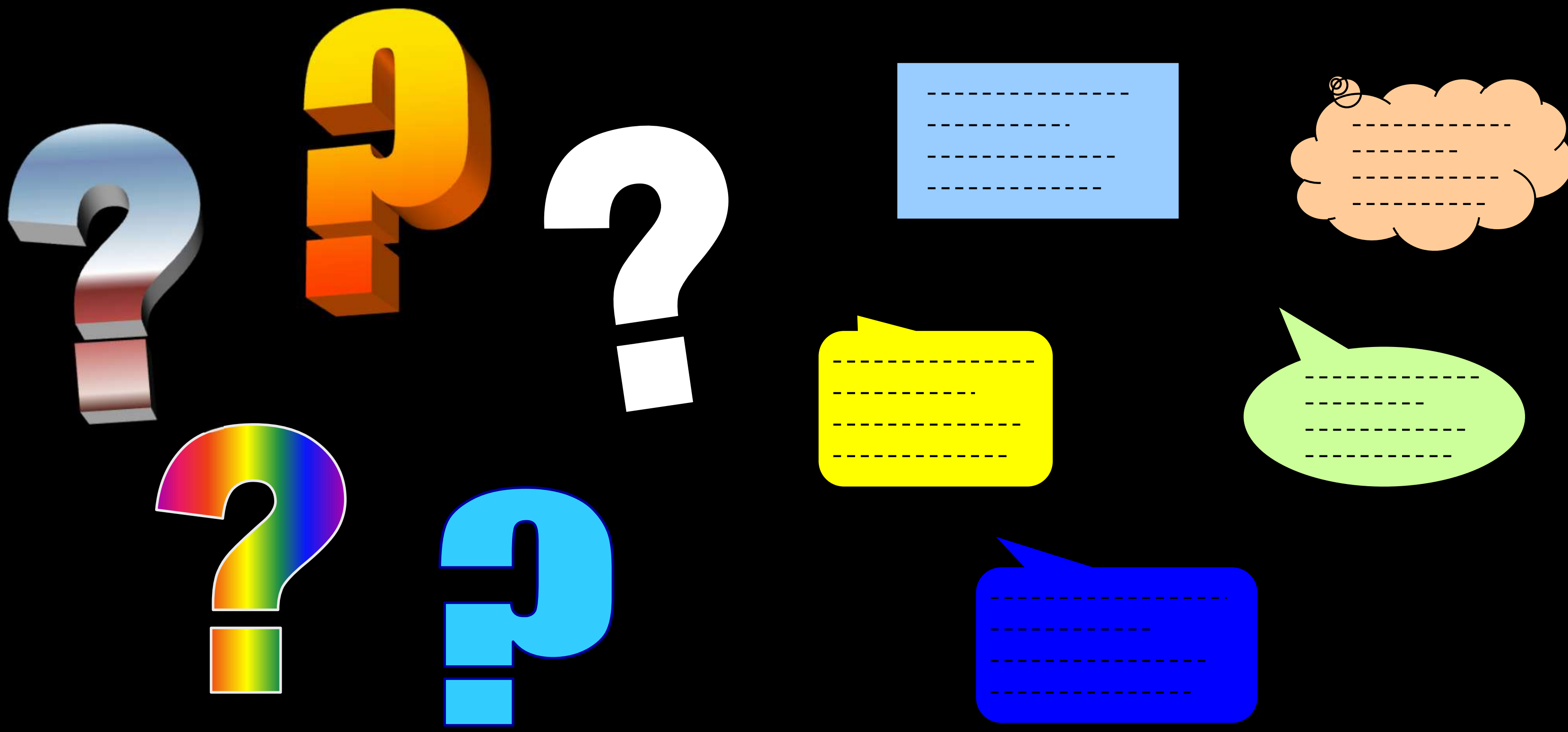

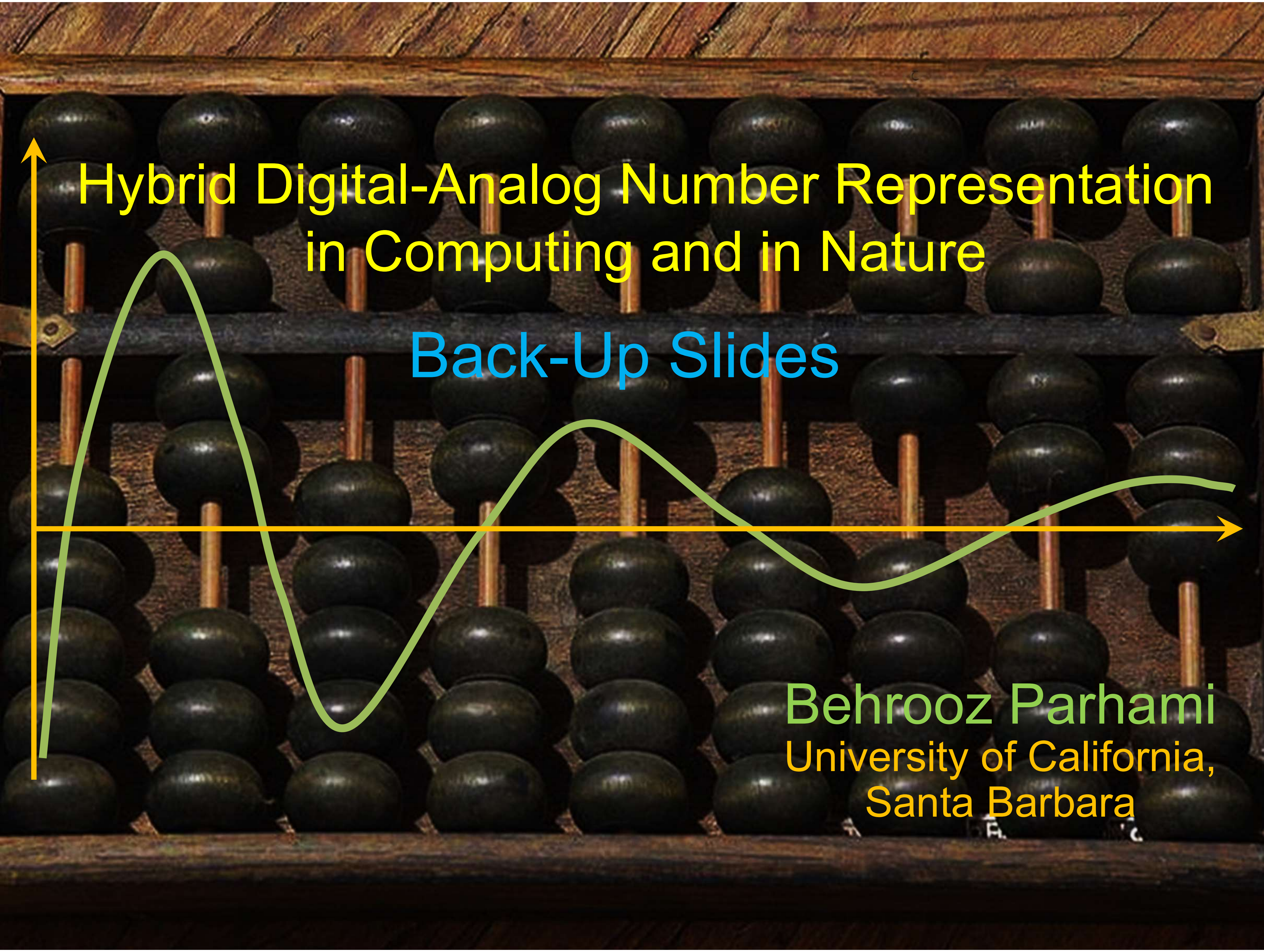

Hybrid Digital-Analog Number Representation in Computing and in Nature

Back-Up Slides

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RNS Dynamic Range

Product M of the k pairwise relatively prime moduli is the *dynamic range*

$M = m_{k-1} \times \ldots \times m_1 \times m_0$

For RNS(8 | 7 | 5 | 3), $M = 8 \times 7 \times 5 \times 3 = 840$

We can take the range of RNS(8|7|5|3) to be [–420, 419] or any other set of 840 consecutive integers

Negative numbers: Complement relative to M

$\langle -x \rangle_{m_i} = \langle M - x \rangle_{m_i}$

$21 = (5 \mid 0 \mid 1 \mid 0)_{\text{RNS}}$

$-21 = (8 - 5 \mid 0 \mid 5 - 1 \mid 0)_{\text{RNS}} = (3 \mid 0 \mid 4 \mid 0)_{\text{RNS}}$

Here are some example numbers in our default RNS(8 | 7 | 5 | 3):

$(0 \mid 0 \mid 0 \mid 0)_{\text{RNS}}$	Represents 0 or 840 or . . .
$(1 \mid 1 \mid 1 \mid 1)_{\text{RNS}}$	Represents 1 or 841 or . . .
$(2 \mid 2 \mid 2 \mid 2)_{\text{RNS}}$	Represents 2 or 842 or . . .
$(0 \mid 1 \mid 3 \mid 2)_{\text{RNS}}$	Represents 8 or 848 or . . .
$(5 \mid 0 \mid 1 \mid 0)_{\text{RNS}}$	Represents 21 or 861 or . . .
$(0 \mid 1 \mid 4 \mid 1)_{\text{RNS}}$	Represents 64 or 904 or . . .
$(2 \mid 0 \mid 0 \mid 2)_{\text{RNS}}$	Represents –70 or 770 or . . .
$(7 \mid 6 \mid 4 \mid 2)_{\text{RNS}}$	Represents –1 or 839 or . . .

RNS Encoding and Arithmetic Operations

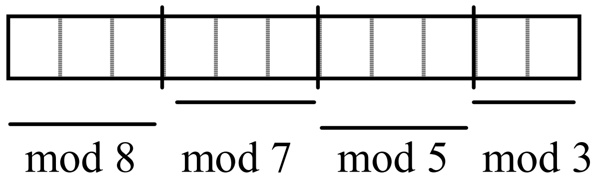


Fig. 4.1 Binary-coded format for RNS(8 | 7 | 5 | 3).

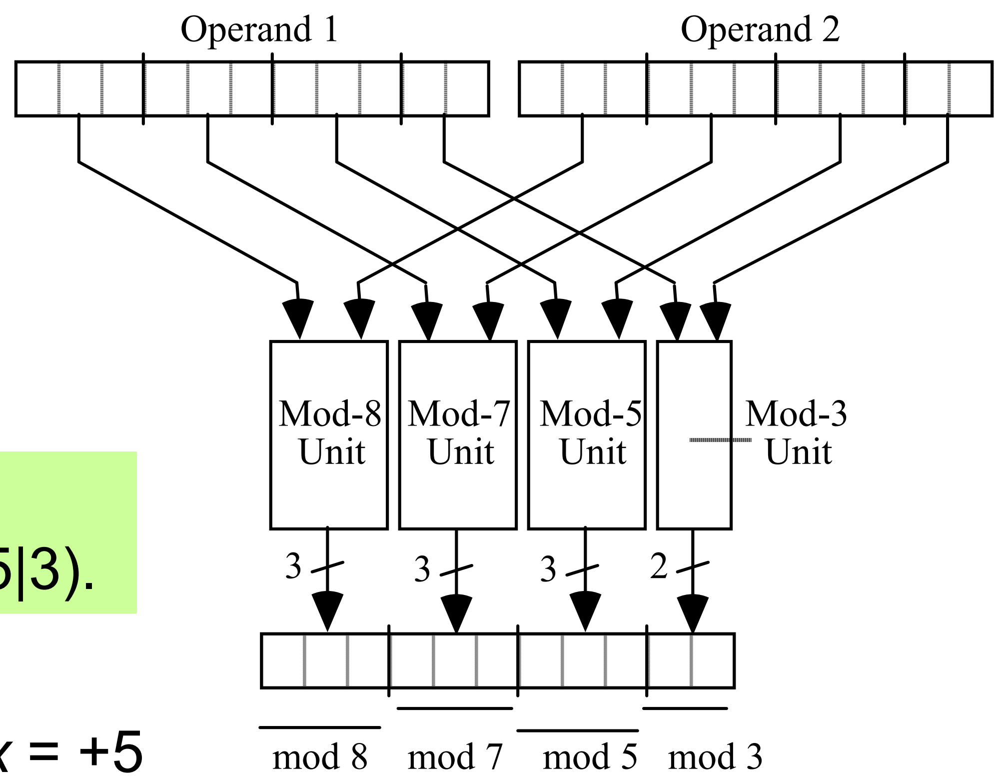


Fig. 4.2 The structure of an adder, subtractor, or multiplier for RNS(8|7|5|3).

Arithmetic in RNS(8 | 7 | 5 | 3)

$(5 \mid 5 \mid 0 \mid 2)_{RNS}$	Represents $x = +5$
$(7 \mid 6 \mid 4 \mid 2)_{RNS}$	Represents $y = -1$
$(4 \mid 4 \mid 4 \mid 1)_{RNS}$	$x + y$: $\langle 5 + 7 \rangle_8 = 4$, $\langle 5 + 6 \rangle_7 = 4$, etc.
$(6 \mid 6 \mid 1 \mid 0)_{RNS}$	$x - y$: $\langle 5 - 7 \rangle_8 = 6$, $\langle 5 - 6 \rangle_7 = 6$, etc. (alternatively, find $-y$ and add to x)
$(3 \mid 2 \mid 0 \mid 1)_{RNS}$	$x \times y$: $\langle 5 \times 7 \rangle_8 = 3$, $\langle 5 \times 6 \rangle_7 = 2$, etc.

Difficult RNS Arithmetic Operations

Sign test and magnitude comparison are difficult

Example: Of the following RNS(8 | 7 | 5 | 3) numbers:

Which, if any, are negative?
Which is the largest?
Which is the smallest?

Assume a range of [–420, 419]

$a = (0 \mid 1 \mid 3 \mid 2)_{RNS}$
$b = (0 \mid 1 \mid 4 \mid 1)_{RNS}$
$c = (0 \mid 6 \mid 2 \mid 1)_{RNS}$
$d = (2 \mid 0 \mid 0 \mid 2)_{RNS}$
$e = (5 \mid 0 \mid 1 \mid 0)_{RNS}$
$f = (7 \mid 6 \mid 4 \mid 2)_{RNS}$

Answers:

$d < c < f < a < e < b$

$-70 < -8 < -1 < 8 < 21 < 64$

Intuitive Justification for CRT

Puzzle: What number has the remainders of 2, 3, and 2 when divided by the numbers 7, 5, and 3, respectively?

$$x = (2 \mid 3 \mid 2)_{RNS(7|5|3)} = (?)_{ten}$$

$$(1 \mid 0 \mid 0)_{RNS(7|5|3)} = \text{multiple of 15 that is 1 mod 7} = 15$$
$$(0 \mid 1 \mid 0)_{RNS(7|5|3)} = \text{multiple of 21 that is 1 mod 5} = 21$$
$$(0 \mid 0 \mid 1)_{RNS(7|5|3)} = \text{multiple of 35 that is 1 mod 3} = 70$$

$$\begin{aligned}(2 \mid 3 \mid 2)_{RNS(7|5|3)} &= (2 \mid 0 \mid 0) + (0 \mid 3 \mid 0) + (0 \mid 0 \mid 2)\\ &= 2 \times (1 \mid 0 \mid 0) + 3 \times (0 \mid 1 \mid 0) + 2 \times (0 \mid 0 \mid 1)\\ &= 2 \times 15 + 3 \times 21 + 2 \times 70\\ &= 30 + 63 + 140\\ &= 233 = 23 \bmod 105\end{aligned}$$

Therefore, $x = (23)_{ten}$

Example RNS with Special Moduli

For RNS(17 | 16 | 15), the weights of the 3 positions are:

2160 3825 2176

Example: $(x_2, x_1, x_0) = (2 \mid 3 \mid 4)_{RNS}$ represents the number

$$\langle 2160\times 2 + 3825\times 3 + 2176\times 4\rangle_{4080} = \langle 24{,}499\rangle_{4080} = 19$$

$2160 = 2^4 \times (2^4 - 1) \times (2^3 + 1) = 2^{11} + 2^7 - 2^4$

$3825 = (2^8 - 1) \times (2^4 - 1) = 2^{12} - 2^8 - 2^4 + 1$

$2176 = 2^7 \times (2^4 + 1) = 2^{11} + 2^7$

$4080 = 2^{12} - 2^4$; thus, to subtract 4080, ignore bit 12 and add 2^4

Reverse converter: Multioperand adder, with shifted x_is as inputs

Limits of Fast Arithmetic in RNS

Known results from number theory

Theorem 4.2: The ith prime p_i is asymptotically $i \ln i$

Theorem 4.3: The number of primes in $[1, n]$ is asymptotically $n / \ln n$

Theorem 4.4: The product of all primes in $[1, n]$ is asymptotically e^n

Implications to speed of arithmetic in RNS

Theorem 4.5: It is possible to represent all k-bit binary numbers in RNS with $O(k / \log k)$ moduli such that the largest modulus has $O(\log k)$ bits

That is, with fast log-time adders, addition needs $O(\log \log k)$ time

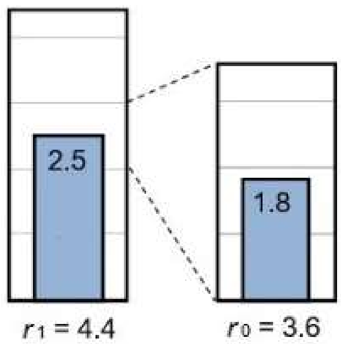
CVNS and CD-RNS Similarities

Two-level scheme: Analog representation at the low (digit) level and digital interpretation at the high (inter-digit) level

CVNS likely has performance edge in general applications

Mixed-radix format: CVNS is based on fixed integer radix, but extension to mixed and non-integer radices is possible

Example mixed-radix CVNS:
Representation of
$9.0 = 2.5 \times 3.6 = 2 \times 3.6 + 1.8$



Approximate computing: Both CVNS and CD-RNS suited to low-precision, adaptive-precision, and lazy arithmetic

CVNS and CD-RNS Differences

Word-level parallelism: CD-RNS has greater affinity with parallel processing of digits in add/subtract/multiply

Input/output overheads: CVNS has simple/direct forward and reverse conversion processes (low-cost and low-energy)
CD-RNS conversions are even more complex than RNS

Noise immunity: Consider 2-digit CVNS and CD-RNS
CVNS range decreases quadratically with increased noise immunity
CD-RNS range decreases linearly with increased noise immunity
(cutting the radix *r* in half, versus using the equation $\mu\varepsilon \le \mu_0\mu_1$)

Fault tolerance: CVNS can be protected through coding
CD-RNS has precision-robustness trade-off built in

CD-RNS Dynamic Range Lower Bound

- CD-RNS with the moduli μ_1 and μ_0
- $\sigma_{-1} = \mu_1$; $\sigma_0 = \mu_0$; $\sigma_{i+1} = \min(|\sigma_{i-1}|_{\sigma_i}, \sigma_i - |\sigma_{i-1}|_{\sigma_i})$
- **Theorem 2:** Dynamic range is at least
  $\mu_0(1 + \lfloor\mu_1/\mu_0\rfloor\lfloor\mu_0/\sigma_1\rfloor)\lfloor\sigma_1/\sigma_2\rfloor\lfloor\sigma_2/\sigma_3\rfloor \ldots \lfloor\sigma_{j-1}/\sigma_j\rfloor$
  where j is the largest index for which $\sigma_j \geq 2\varepsilon_{max}$
- **Intuition:** Remove floors to get $\mu_0\mu_1/(2\varepsilon_{max})$
- **Example 6:** CD-RNS with $\mu_1 = 4.4$, $\mu_0 = 3.6$, $\varepsilon_{max} = 0.2$
  $\Rightarrow$ $\sigma_1 = 0.8$, $\sigma_2 = 0.4$ $\Rightarrow$ Dynamic range ≥ 36.0

CD-RNS Dynamic Range Upper Bound

- CD-RNS with the moduli μ_1 and μ_0
- δ = Largest number that divides μ_1 and μ_0 if it exists, 0 otherwise
- **Theorem 3:** Dynamic range is at most $\max(\mu_0\lfloor\mu_1/\gamma\rfloor, \mu_1\lfloor\mu_0/\gamma\rfloor)$ where $\gamma = \max(2\varepsilon_{max}, \delta)$
- **Intuition:** Remove floors to get $\mu_0\mu_1/\gamma$
- **Example 6:** CD-RNS with $\mu_1 = 4.4$, $\mu_0 = 3.6$, $\varepsilon_{max} = 0.2$ $\Rightarrow$ $\delta = 0.4$, $\gamma = 0.4$ $\Rightarrow$ Dynamic range ≤ 39.6

CD-RNS Lower/Upper Bounds Example

- Example 10 in paper
- Fix μ_1 at 4.4
- Vary μ_0 in steps of 0.1
- Range varies (dashed)
- Tightness varies
- Matching of upper bound = Optimality?
- Achieving wider range