Hexagonal and Pruned Torus Networks as Cayley Graphs

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Abstract

Hexagonal mesh and torus, as well as honeycomb and certain other pruned torus networks, are known to belong to the class of Cayley graphs which are node-symmetric and possess other interesting mathematical properties. In this paper, we use Cayley-graph formulations for the aforementioned networks, along with some of our previous results on subgraphs and coset graphs, to draw conclusions relating to internode distance and network diameter. We also use our results to refine, clarify, and unify a number of previously published properties for these networks and other networks derived from them.

Keywords– Cayley digraph, Coset graph, Diameter, Distributed system, Hex mesh, Homomorphism, Honeycomb mesh or torus, Internode distance.

1. Introduction

The fact that Cayley (di)graphs and coset graphs are excellent models for interconnection networks, of the types studied in relation to parallel processing and distributed computation, is widely acknowledged [1], [2], [4]. Many well-known, and practically significant, interconnection networks are Cayley (di)graphs or coset graphs. For example, hypercube (binary *q*-cube), butterfly, and cube-connected cycles networks are Cayley graphs, while de Bruijn and shuffle-exchange networks are instances of coset graphs [4], [11].

Much work on interconnection networks can be categorized as ad hoc design and evaluation. Typically, a new interconnection scheme is suggested and shown to be superior to some previously studied network(s) with respect to one or more performance or complexity attributes. Whereas Cayley (di)graphs have been used to explain and unify interconnection networks with many ensuing benefits, much work remains to be done. As suggested by Heydemann [4], general theorems are lacking for Cayley digraphs and more group theory has to be exploited to find properties of Cayley digraphs.

In this paper, we explore the relationships between Cayley (di)graphs and their subgraphs and coset graphs with respect to subgroups and formulate general results on homomorphism and diameter formulas between them. We provide several applications of these results to well-known interconnection networks such as hexagonal torus, honeycomb torus, and several other classes of pruned torus networks.

Before proceeding further, we introduce some definitions and notations related to (di)graphs in general, Cayley (di)graphs in particular, and to interconnection networks. For more definitions and mathematical results on graphs and groups we refer the reader to [3], for instance, and on interconnection networks to [6], [8]. Unless noted otherwise, all graphs in this paper are undirected. Proofs are omitted here and will be provided in a more complete version of this paper.

A digraph $\Gamma = (V, E)$ is defined by a set V of vertices and a set E of arcs or directed edges. The set E is a subset of elements (u, v) of $V \times V$. If the subset E is symmetric, that is, $(u, v) \in E$ implies $(v, u) \in E$, we identify two opposite arcs (u, v) and (v, u) by the undirected edge (u, v). Because we deal with undirected graphs in this paper, no problem arises from using the same notation (u, v) for a directed arc from u to v or an undirected edge between u and v.

Let G be a (possibly infinite) group and S a subset of G. The subset S is said to be a generating set for G, and the elements of S are called generators of G, if every element of G can be expressed as a finite product of their powers. We also say that G is generated by S. The Cayley digraph of the group G and the subset S, denoted by $Cay(G, S)$, has vertices that are elements of G and arcs that are ordered pairs (g, gs) for $g \in G$, $s \in S$. If S is a generating set of G then we say that $Cay(G, S)$ is the Cayley digraph of G generated by S. If $1 \notin S$ (where 1 is the identity element of G) and $S = S^{-1}$, then $Cay(G, S)$ is a simple graph.

Assume that Γ and Σ are two digraphs. The mapping ϕ of $V(\Gamma)$ to $V(\Sigma)$ is a homomorphism from Γ to Σ if for any $(u, v) \in E(\Gamma)$ we have $(\phi(u), \phi(v)) \in E(\Sigma)$. In particular, if ϕ is a bijection such that both ϕ and the inverse of ϕ are homomorphisms then ϕ is called an isomorphism of Γ to Σ. Let G be a (possibly infinite) group and S a subset of G. Assume that K is a subgroup of G (denoted as $K \leq G$). Let ${}^{G}/_{K}$ denote the set of the right cosets of K in G. The (right) coset graph of G with respect to subgroup K and subset S, denoted by $Cos(G, K, S)$, is the digraph with vertex set ${}^{G}/_{K}$ such that there exists an arc (Kg, Kg') if and only if there exists $s \in S$ and $Kgs = Kg'$.

The following basic theorem, which can be easily proven, is helpful in establishing some of our subsequent results [12].

Theorem 1. For $g \in G$, $S \subseteq G$, and $K \leq G$, the mapping ϕ: $g \to Kg$ is a homomorphism from $Cay(G, S)$ to $Cos(G, K, S)$. ■

2. An Inequality for Diameter

For a digraph Ω, $D(\Omega)$ denotes the diameter of Ω, that is, it is the longest distance between vertices of Ω. Now let G be a finite group and $K \leq G$. Assume that $\Gamma = Cay(G, S)$ and $\Delta = Cos(G, K, S)$ for some generating set S of G. Let $D(\Gamma_K)$ denote the longest distance between vertices of K in Γ. Similar to Theorem 2 in [11] we have the following result.

Theorem 2. $D(\Gamma) \leq D(\Delta) + D(\Gamma_K)$. ■

We can apply Theorem 2 to some well-known interconnection networks. Although many results on these interconnection networks are known, the unified treatment is still beneficial.

Example1. Diameter of hypercube network. We know that the hypercube $Q_q = Cay(Z_2^q, S)$, where $S = \{0^{(i-1)}10^{(q-i)} \mid i = 1, \ldots, q\}$. Let $K = Z_2$. Then we have $\Delta = Cos(Z_2^q, Z_2, S)$, leading to $D(Q_q) \leq D(\Delta) + 1$ by Theorem 2. But $\Delta \cong Q_{q-1}$ and thus $D(\Delta) \leq q - 1$ by induction. Therefore, $D(Q_q) \leq q$. Since $dis(0^{(q)}, 1^{(q)}) = q$, we have $D(Q_q) = q$. ■

Example 2. Relating the butterfly network BF_q to de Bruijn network DB_2^q. Let $N = Z_2^q$ and $K = Z_q$. Then, $G = Z_2 \; wr \; Z_q$ is a semidirect product of N by K. Assuming $S = \{0^{(q)}1, 0^{(q-1)}11\}$, from [11] we have $\Gamma = Cay(G, S) = BF_q$ and $\Delta = Cos(G, K, S) = DB_2^q$. Since $D(DB_2^q) = q$, we obtain $D(BF_q) \leq q + \lfloor q/2 \rfloor$ by Theorem 2. In fact it is verified easily that $D(BF_q) = q + \lfloor q/2 \rfloor$. ■

3. Hexagonal Torus Networks

Let $G = Z \times Z$, with Z the infinite cyclic group of integers, and consider $\Gamma = Cay(G, S)$ with $S = \{(\pm1, 0), (0, \pm1), (1, 1), (-1, -1)\}$. It is evident that Γ is isomorphic to the hexagonal mesh network [7], [10]. Figure 1 depicts a small part of an infinite hexagonal mesh network in which the six neighbors of the center node (0, 0) are shown. A finite hexagonal mesh is obtained by simply using the same connectivity rules for a finite subset of the nodes located within a regular boundary (often a rectangle or hexagon). In the latter case, wraparound links are sometimes provided to keep the node degree uniformly equal to 6. Here, we do not concern ourselves with these variations and deal mainly with the hexagonal torus networks.

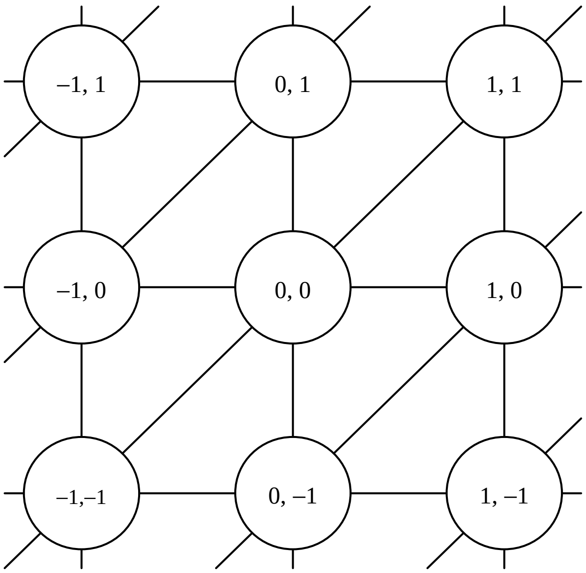

Fig. 1. Connectivity pattern for hexagonal mesh network, where node (i, j) is connected to the six neighboring nodes ($i \pm 1$, j), (i, $j \pm 1$), ($i + 1$, $j + 1$), and ($i - 1$, $j - 1$).

Let $H = Z_l \times Z_k$, where Z_l and Z_k are cyclic groups of orders l and k respectively(l and k are both positive integers). Assume that S is defined as above. Then $\Delta = Cay(H, S)$ is the hexagonal torus of order lk. Let $K = \langle l\rangle \times \langle k\rangle$. Then $\Delta \cong$ Cos($Z \times Z$, K, S) and so the hexagonal torus is a homomorphic image of the infinite hexagonal mesh according to Theorem 1.

Using the results obtained for infinite hexagonal meshes, we may deal with problems on hexagonal tori which are, in general, more difficult. Let Δ be defined as above. Then we have the following result.

Proposition 1. For the hexagonal torus Δ of order lk and integers a and b, $0 \le a < l$, $0 \le b < k$, we have $dis((0, 0), (a, b)) = min(max(a, b), max(l - a, k - b), l - a + b, k + a - b)$. ■

4. Honeycomb and Other Tori

Let G be a (possibly infinite) group and S a subset of G. Consider the problem of constructing a group G'' and its generating set S'' such that $G'' = G$ as sets and $S'' \subseteq S$, and a homomorphism $\phi: \Gamma'' \to \Gamma$, where $\Gamma = Cay(G, S)$ and $\Gamma'' = Cay(G'', S'')$. It is shown in [12] that a number of pruning schemes, including the one studied in [9], are equivalent to the construction above. Pruning of interconnection networks constitutes a way of deriving variants with lower cost and greater scalability [5]. If pruning is done with care, and in a systematic fashion, many of the desirable properties of the original (unpruned) network, including symmetry and regularity, can be maintained while reducing both the node degree and wiring density. We give new proofs of the construction above in the following example.

Example 3. Pruned three-dimensional toroidal network T_1 of [5]. Let $G = (\langle a\rangle \langle b\rangle) \langle c\rangle$ be the group generated by the elements a, b, c, satisfying the relations $a^k = b^k = c^k = 1$, $ab = ba$, $c^{-1}ac = b^{-1}$, $c^{-1}bc = a^{-1}$. Here, k is even. Thus the group $\langle a\rangle\langle b\rangle = \langle a, b\rangle$ is a direct product of $\langle a\rangle$ and $\langle b\rangle$, and G is a semidirect product of $\langle a, b\rangle$ by $\langle c\rangle$. Let $S = \{a, a^{-1}, c, c^{-1}\}$ and $\Delta_1 = Cay(G, S)$. We now prove that Δ_1 is isomorphic to the pruned 3D toroidal network T_1 in [5], as shown in Fig. 2. In fact, let $a_1 = (1, 0, 0)^T$, $b_1 = (0, 1, 0)^T$, $c_1 = (0, 0, 1)^T$. It is easily shown that a_1, b_1, and c_1 satisfy the same relations as those of Δ_1; namely, $a_1^k = b_1^k = c_1^k = 1$, $a_1b_1 = b_1a_1$, $c_1^{-1} a_1c_1 = b_1^{-1}$, $c_1^{-1} b_1c_1 = a_1^{-1}$. Thus, the mapping $a \to a_1$, $b \to b_1$, $c \to c_1$ is an isomorphism of Δ_1 to T_1. ■

Example 4. Pruned three-dimensional toroidal network T_2 of [5], depicted in Fig. 3. We obtain the results for the network T_2 in a manner similar to those for T_1 of Example 3. Let $G = \langle a, b\rangle\langle c\rangle$ be the group generated by the elements a, b, c, satisfying the relations $a^{2k} = b^{2k} = c^k = 1$, $a^2 = b^2$, $(ab)^{k/2} = (ba)^{k/2} = 1$, $c^{-1}ac = b$, $c^{-1}bc = a$. Here k is even and $\langle a, b\rangle = \langle ab\rangle\langle a\rangle$ is a complex group. Let $S = \{a, a^{-1}, c, c^{-1}\}$ and $\Delta_2 = Cay(G, S)$. Then, the mapping $a \to (1, 0, 0)^T$, $b \to (0, -1, 0)^T$, $c \to (0, 0, 1)^T$ is an isomorphism of Δ_2 to T_2. ■

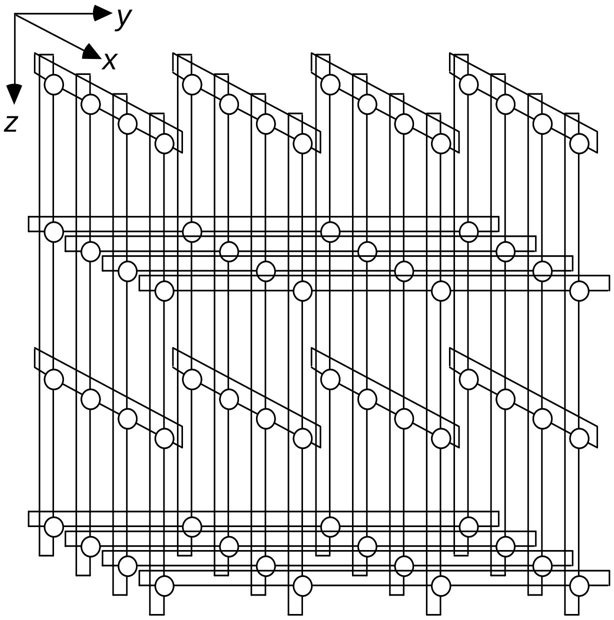

Fig. 2. Pruned 3D torus network (T_1 of [5]).

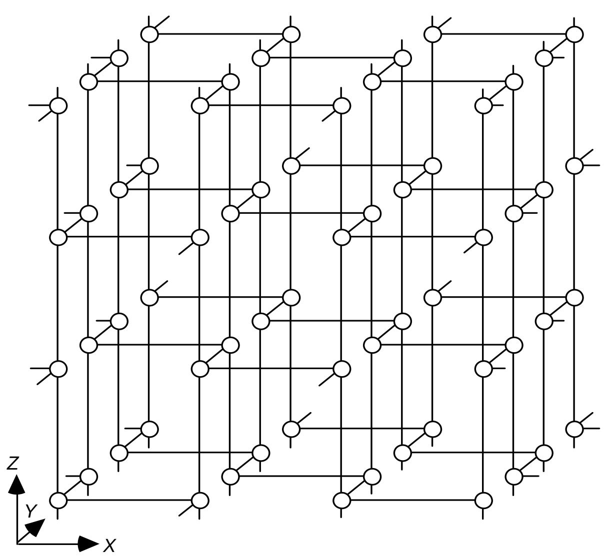

Fig. 3. Pruned 3D torus network (T_2 of [5]). To avoid clutter, the wraparound connections are not drawn fully.

The authors of [9] studied the honeycomb torus network as a pruned 2D torus. They also proved that the honeycomb torus network is a Cayley graph, without explicating its associated group. We filled this gap in [12], while also showing why the parameter k in [9] must be even. Let $G = (\langle c\rangle\,\langle b\rangle)\,\langle a\rangle$ be the group generated by the elements a, b, c, satisfying the relations $a^k = b^2 = c^{l/2} = 1$, $bcb = c^{-1}$, $aba^{-1} = c^{-1}b$, $aca^{-1} = c^{-1}$. Here, k and l are even integers. Thus, the group $\langle c\rangle\,\langle b\rangle = \langle c, b\rangle$ is a semidirect product of $\langle c\rangle$ by $\langle b\rangle$, and G is a semidirect product of $\langle c, b\rangle$ by $\langle a\rangle$. Let $S = \{a, a^{-1}, b\}$ and $\Delta = Cay(G, S)$. We have shown in [12] that Δ is isomorphic to the honeycomb torus network in [9].

Proposition 2. In [12], we introduce the infinite honeycomb network as a Cayley graph of a different infinite group. Let $G = (\langle c\rangle\ \langle b\rangle)\ \langle a\rangle$, where $\langle c\rangle$ and $\langle a\rangle$ are infinite cyclic groups, and c, b, a satisfy the relationships $b^2 = 1$, $bcb = c^{-1}$, $aba^{-1} = c^{-1}b$, $aca^{-1} = c^{-1}$. Let $S = \{a, a^{-1}, b\}$ and $\Delta_\infty = Cay(G, S)$. Then Δ_∞ is isomorphic to the infinite honeycomb network (see Fig. 4). ■

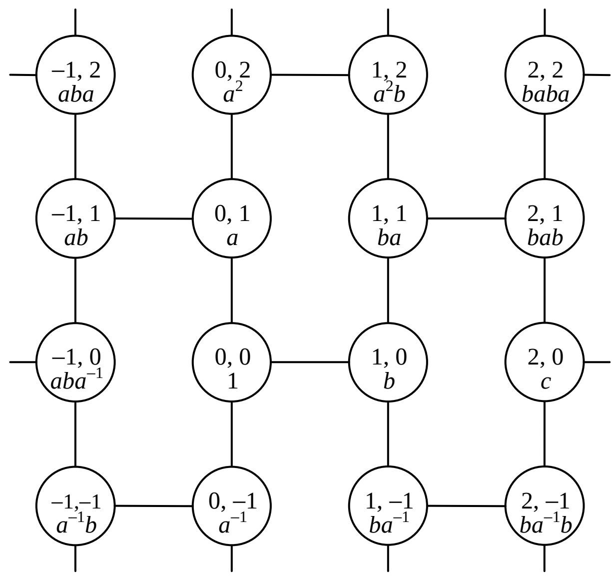

Fig. 4. Connectivity pattern for honeycomb mesh network. Each node is labeled in two ways corresponding to its coordinates on the grid (upper label) and the notation in Proposition 2 (lower label), with the associations being (0, 1) = *a*, (1, 0) = *b*, (2, 0) = *c*.

Now let $N = \langle a^k\rangle\,\langle c^{l/2}\rangle$, where k and l are even integers. We can easily verify that $N \lhd G$ (N is a normal subgroup of G). Construct the quotient group $G' = {}^{G}/_{N}$ and let $S' = \{Na, Na^{-1}, Nb\}$; the graph $Cay(G', S')$ is isomorphic to honeycomb torus network. Thus the honeycomb torus is a homomorphic image of the infinite honeycomb network by Theorem 1.

For the infinite honeycomb network Δ_∞ any element of G can be expressed as the product $c^j b^l a^i$, where l is 0 or 1 and j and i are integers. We obtained in [12] the distance formula between vertices 1 (the identity of G) and $c^j b^l a^i$, as stated in the following theorem.

Theorem 3. For $|i| \le |2j + l|$, we have $dis(1, c^j b^l a^i) = |4j + l + ½ [(-1)^{i+l} - (-1)^l]|$; otherwise, $dis(1, c^j b^l a^i) = |i| + |2j + l|$. ■

In [12] we proved that Theorem 3 has the following corollary.

Corollary 1. In the infinite honeycomb network, distance between nodes (x, y) and (u, v) is obtained as follows. If $|v - y| \le |u - x|$, then $dis((x, y), (u, v))$ is given by $|2(u - x) + ½[(-1)^{u+v} - 1]|$, when $x + y \equiv 0 \bmod 2$, and by $|2(x - u) + ½ [(-1)^{u+v+1} - 1]|$, otherwise. In the remaining case of $|v - y| \ge |u - x|$, we have $dis((x, y), (u, v)) = |u - x| + |v - y|$. ■

Using Theorem 3 and Corollary 1, we obtain a result on the diameter of honeycomb torus network Δ, generalizing theorem 3 in [9] which states that $D(\Delta) = l$ when $l = k$.

Theorem 4. For the honeycomb torus network Δ, we have $D(\Delta) = \max(l, (l + k)/2)$. ■

As an application of our construction, we consider the pruned three-dimensional toroidal networks T_1 of Example 3 further. We shall derive a formula for the distance between the identity element 1 and the vertex $a^i b^j c^l$, where $0 \le i, j, l < k$. Theorem 3 of [5], characterizing the network T_1, is a direct corollary of this formula.

Theorem 5. For T_1 we have $dis(1, a^i b^j c^l) = min(i, k - i) + min(j, k - j) + min(l, k - l)$ if $l > 0$. When $l = 0$, we have $dis(1, a^i b^j) = min(i, k - i) + min(j, k - j) + 2$ if $j > 0$ and $min(i, k - i)$ otherwise. ■

Corollary 2. For the network T_1 with $k \ge 4$, we have $D(T_1) = 3k/2$. ■

Finally we shall show that Theorem 2 in [5] does not hold in general. The following example shows that the pruned three-dimensional toroidal network T_1 of Example 3 may not be edge-symmetric.

Example 5. Consider the case of $k = 4$. Let $A = Aut(T_1)$ be the automorphism group of T_1. We shall show that there is no $\sigma \in$ A such that $\sigma(1) = 1$ and $\sigma(c) = a$. Similarly, we show that there is no $\tau \in A$ such that $\tau(1) = a$ and $\tau(c) = 1$. Hence T_1 is not edge-symmetric for $k = 4$. In fact, we show that the assumption $\sigma \in A$, such that $\sigma(1) = 1$ and $\sigma(c) = a$, leads to a contradiction. Since the edge $(1, c)$ is in the cycle $C = \{1, c, c^2, c^3\}$ and the edge $(1, a)$ is only in the cycle $A' = \{1, a, a^2, a^3\}$, C is mapped to A' by σ. Hence, $\sigma(c^2) = a^2$ and $\sigma(c^3) = a^3$. Now consider the cycle $B = \{a, ac, ac^2, ac^3\}$. Since $\sigma(1) = 1$ and $(1, a)$ is an edge, $(1, \sigma(a))$ is also an edge. This implies that $\sigma(a)$ equals c or c^{-1}. Let $\sigma(a) = c$ (the case of $\sigma(a) = c^{-1}$ is similar). Since (a, ac) is an edge, $(c, \sigma(ac))$ is also an edge. Because the cycle B cannot be mapped to the cycle C by σ, we have $\sigma(ac) = ca$ or ca^{-1}. Given that (c^2, c^2a) is an edge and $\sigma(c^2) = a^2$, $(a^2, \sigma(c^2a))$ is also and edge. Therefore, $\sigma(c^2a)$ equals a^2c or a^2c^{-1}. Since $c^2a = ac^2$, $\{a, ac, c^2a\}$ is in the cycle B. But $\{\sigma(a), \sigma(ac), \sigma(c^2a)\}$ is not in any cycle of order four. This is a contradiction. ■

5. Conclusion

In this paper, we have provided a number of general results on homomorphism and diameter between Cayley (di)graphs and their subgraphs and coset graphs. We have also demonstrated the applications of these results to some well-known interconnection networks, including hexagonal and honeycomb tori and related networks.

We are currently investigating the applications of our method to the problems related to routing and average internode distance in certain subgraphs of honeycomb networks. We also aim to extend our results to other classes of networks as well as to other topological properties of networks.

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