Comments

Comments on "Low Diameter Interconnections for Routing in High-Performance Parallel Systems," with Connections and Extensions to Arc Coloring of Coset Graphs

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Abstract—Recently, Melhem presented a "new" class of low-diameter interconnection (LDI) networks in this journal [10]. We note that LDI networks are the same as the previously known generalized de Bruijn graphs, point out an error in the decomposition of LDI networks into permutations, and find that the correct decomposition scheme is an instance of arc coloring for coset graphs. Hence, we pursue a number of general results on arc coloring of coset graphs that can be applied to this particular decomposition problem as well as within many other contexts, including complete arc coloring and normality of coset graphs.

Index Terms—Arc coloring, Cayley graph, connected regular digraph, coset graph, generalized de Bruijn digraph, group, network isomorphism.

1 Introduction

Cayley (di)graphs and coset graphs have been advanced and extensively studied as interconnection network models for parallel processing [1], [2], [6]. For example, the well-known hypercube, butterfly, cube-connected cycles, and honeycomb networks are Cayley graphs, while de Bruijn and shuffle-exchange networks are coset graphs [5], [6], [11], [13], [14], [15], [16], [17]. Throughout the history of parallel processing, many interconnection schemes have been rediscovered by researchers who were either unaware of equivalent existing networks or else, due to notational differences and the ad hoc nature of much of the work in the field, could not relate their discoveries to prior efforts. This situation would improve immensely if foundational studies were given more attention and if Heydemann's suggestion that more group theory be used to find properties of Cayley digraphs and coset graphs [6] were followed.

Melhem's recent introduction of a class of low-diameter interconnection (LDI) networks [10] constitutes an example of the rediscoveries just mentioned. In this correspondence, we show that LDI networks are isomorphic to the previously known generalized de Bruijn graphs and point out an error in the method offered for decomposing LDI networks into permutations. Noting that our corrected decomposition scheme is an instance of arc coloring for coset graphs, we pursue a number of general results on arc coloring of coset graphs that can be applied to this particular decomposition problem as well as within many other contexts, including complete arc coloring and normality of coset graphs.

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2 Background and Definitions

Before presenting our main points, we need some terminology and notational conventions. Assume that $\Gamma = (V, E)$ is a connected d-regular digraph, with loops or (i, i) arcs allowed, and let $C = \{c_1, \ldots, c_d\}$ be the set of arc colorings of Γ, where $V = \{1, 2, \ldots, n\}$ is the vertex set and E is the arc set of Γ. Elements of C are permutations on V such that $c_i(x) = y$ iff (x, y) is an arc of Γ with color c_i (see [5]). Now, let $P = \langle c_1, \ldots, c_d \rangle$ be the permutation group generated by C and $Cos(P, P_1, C)$ be the coset graph of P with respect to P_1 and C, where P_1 is the stabilizer of 1, the identity element in P. Then, we have $\Gamma \cong Cos(P, P_1, C)$.

Assume that $P = P_1 g_1 \cup \ldots \cup P_1 g_n$ is a decomposition of right cosets of P with respect to P_1 such that $1^{g_i} = i$, $i = 1, 2, \ldots, n$. Then, the vertex set of $Cos(P, P_1, C)$ is $\{P_1 g_1, \ldots, P_1 g_n\}$ and the arc set is $\{(P_1 g_i, P_1 g_i c) \mid c \in C, i = 1, 2, \ldots, n\}$. It is easily shown that $S_n = (S_n)_1 g_1 \cup \ldots \cup (S_n)_1 g_n$ is a decomposition of right cosets of S_n with respect to $(S_n)_1$, where S_n is the symmetric group on V.

Informally, decomposition of an n-vertex d-regular digraph into a set of d permutations means identifying disjoint subsets E_i of arcs such that $E = \cup E_i$, each subset E_i contains n arcs, and every node has one incoming and one outgoing arc in any E_i. Such network decompositions are of interest because the subsets correspond to conflict-free routing steps with maximal throughput, under the assumption that each node can handle one incoming and one outgoing message at a time (single-port communication). They also point to the network's realizability by permutation networks. Note that if we associate each subset E_i with a color, the preceding decomposition corresponds to coloring of arcs using d different colors so that each incoming (outgoing) arc for any given vertex has a distinct color.

We refer the reader to [4] for additional background on graph theory and group theory and to [9] and [12] for key notions of interconnection networks.

3 LDI Networks and Their Decomposition

An n-node, d-regular LDI network [10], LDI(n, d), has its nodes numbered 0 to $n-1$, with an arc leading from node i to node j iff $j = id + a \bmod n$, for some a in $\{0, 1, \ldots, d-1\}$. LDI networks, presented as "new" interconnections in [10], correspond precisely to generalized de Bruijn digraphs [3], [7], [8]. Moreover, the decomposition of LDI networks into permutations is incorrectly specified in [10]. For example, by (2) of [10], which becomes $c_k = \{\langle i, id + a \bmod n \rangle \mid i = 0, 1, \ldots, n-1$ and a satisfying $k = \lfloor i/d \rfloor + a \bmod d\}$ using our notation, the 0th permutation is $c_0 = \{\langle 0, 0 \rangle, \langle 3, 0 \rangle, \ldots\}$ for LDI(7, 2) and $c_0 = \{\langle 1, 4 \rangle, \langle 8, 4 \rangle, \ldots\}$ for LDI(10, 4). These c_0 sets clearly do not represent permutations.

To derive a correct decomposition, let $h = \gcd(n, d)$, the greatest common divisor of n and d. Also, let $i = (n/h)t + r$, where $t \in \{0, 1, \ldots, h-1\}$ and $r \in \{0, 1, \ldots, n/h - 1\}$. Then, the d permutations c_k, $k = 0, 1, \ldots, d-1$, may be defined as:

$$c_k = \{\langle i, id + k - t \bmod n \rangle | i = 0, 1, \ldots, n-1\}.$$

In fact, for $0 \le a < b < h$ and $0 \le r_a, r_b < n/h$, we have $(r_b - r_a)d \neq (b - a) \bmod n$ and, thus, $(na/h + r_a)d + k - a \neq (nb/h + r_b)d + k - b \bmod n$. Thus, our revised c_k do represent permutations.

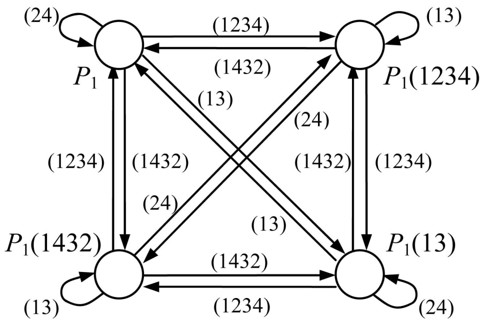

Fig. 1. $Cos(P, P_1, D)$, with $P = \langle(1234)\rangle\,\langle(13)\rangle$, $P_1 = \langle(24)\rangle$, $D = \{(1234), (1432), (13), (24)\}$.

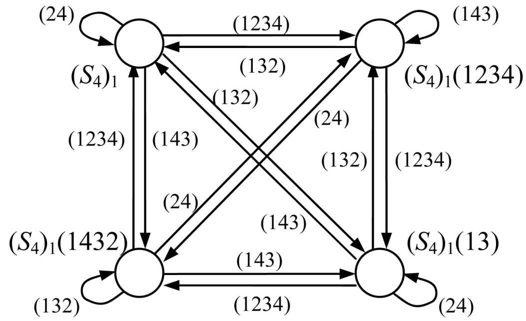

Fig. 2. $Cos(S_4, (S_4)_1, C)$, with $C = \{(1234), (132), (143), (24)\}$, is a complete coloring of the digraph in Fig. 1.

Note that $\{c_k \mid k = 0, \ldots, d-1\}$ is an arc coloring of $\mathrm{LDI}(n, d)$, that is, a mapping that assigns one of d colors to each arc such that the arcs incoming to or outgoing from any vertex have different color assignments. This motivates us to look at arc colorings for d-regular digraphs.

4 Arc Coloring of Coset Graphs

We start by proving the following result:

Lemma 1. *The coset graph of S_n with respect to $(S_n)_1$ and C, viz. $Cos(S_n, (S_n)_1, C)$, is isomorphic to $\Gamma = Cos(P, P_1, C)$.*

Proof. Define the correspondence f: $P_1 g_i \rightarrow (S_n)_1 g_i$, $i = 1, 2, \ldots, n$. Clearly, f is a bijective mapping. Let $P_1 g_i c = P_1 g_j$ for some j. Then, f: $P_1 g_j \rightarrow (S_n)_1 g_j = (S_n)_1 g_i c$ and, so, f preserves adjacencies. Similarly, we can prove the f^{-1} also preserves adjacencies. The conclusion that f is an isomorphism from Γ to $Cos(S_n, (S_n)_1, C)$ follows. □

By Lemma 1, we know that any connected regular digraph Γ can be expressed as a coset graph of the form $Cos(S_n, (S_n)_1, C)$. Now, let $S = (S_n) \setminus \bigcup_{i-1}^{n} (S_n)_i$. Then, we have the following:

Lemma 2. *A coset graph $\Sigma = Cos(S_n, (S_n)_1, C)$ is isomorphic to a connected regular digraph $\Gamma = Cos(P, P_1, C)$ with $P = \langle C \rangle$ iff Σ is connected and $CC^{-1} \subseteq 1 \cup S$.*

Proof. Assume that $\Sigma = Cos(S_n, (S_n)_1, C) \cong \Gamma$, where $\Gamma = Cos(P, P_1, C)$ is a connected regular digraph with $P = \langle C \rangle$. Then, $P_1 g_i c \neq P_1 g_i c_1$, where $c, c_1 \in C$ and $c \neq c_1$. Consequently, by virtue of $i = 1^{g_i}$, $cc_1^{-1} \notin \bigcup_{i-1}^{n} g_i^{-1} P_1 g_i = \bigcup_{i-1}^{n} P_i$, and thus $cc_1^{-1} \in S$. Hence, $CC^{-1} \subseteq 1 \cup S$. Conversely, if Σ is connected and $CC^{-1} \subseteq 1 \cup S$, then $(S_n)_i c \neq (S_n)_i c_1$ and, so, $P_i c \neq P_i c_1$, where $c, c_1 \in C$ and $c \neq c_1$. Thus, by Lemma 1, Σ is isomorphic to the connected regular digraph $\Gamma = Cos(P, P_1, C)$, where $P = \langle C \rangle$. □

According to Lemma 2, in order to study connected regular digraphs, we only need to study special kinds of connected coset graphs $Cos(S_n, (S_n)_1, C)$ with $CC^{-1} \subseteq 1 \cup S$. In the discussion to follow, all regular digraphs considered are connected.

Now, suppose that $\Gamma = Cos(S_n, (S_n)_1, D)$ and $\Sigma = Cos(S_n, (S_n)_1, C)$ are two regular digraphs, where $DD^{-1}, CC^{-1} \subseteq 1 \cup S$. Consider the mapping σ: $(S_n)_1 g_i \rightarrow (S_n)_1 g_{\sigma(i)}$, where $\sigma(i) = i^{\sigma}$, $i = 1, 2, \ldots, n$. Then, $(S_n)_1 g_{\sigma(i)} = (S_n)_1 g_i \sigma$ by $1^{g_i} = i$ and we can prove the following:

Theorem 1. *The mapping σ is an isomorphism from Γ to Σ iff $(S_n)_i D = (S_n)_i \sigma C \sigma^{-1}$ for $i = 1, 2, \ldots, n$.*

Proof. Consider the mapping σ: $(S_n)_1 g_i \rightarrow (S_n)_1 g_{\sigma(i)}$, $i = 1, 2, \ldots, n$. Assume that $(S_n)_1 g_i d = (S_n)_1 g_k$, $d \in D$. Then, σ preserves adjacency iff there exists $c_i \in C$ such that $(S_n)_1 g_{\sigma(k)} = (S_n)_1 g_{\sigma(i)} c_i$, that is, $(S_n)_1 g_k \sigma = (S_n)_1 g_i \sigma c_i$, since $(S_n)_1 g_{\sigma(i)} = (S_n)_1 g_i \sigma$. This yields $(S_n)_1 g_i d = (S_n)_1 g_i \sigma c_i \sigma^{-1}$. Thus, we find that σ preserves adjacency iff $(S_n)_1 g_i D \subseteq (S_n)_1 g_i \sigma C \sigma^{-1}$, that is, iff $(S_n)_i D \subseteq (S_n)_i \sigma C \sigma^{-1}$. Similarly, we can prove that σ^{-1} preserves adjacency iff $(S_n)_i \sigma C \sigma^{-1} \supseteq (S_n)_i D$ for $i = 1, 2, \ldots, n$. Consequently, σ is an isomorphism from Γ to Σ iff $(S_n)_i D = (S_n)_i \sigma C \sigma^{-1}$ for $i = 1, 2, \ldots, n$. □

Corollary 1. *The mapping σ is an automorphism of Σ iff $(S_n)_i C = (S_n)_i \sigma C \sigma^{-1}$ for $i = 1, 2, \ldots, n$.*

Let $G = Aut(\Sigma)$ be the automorphism group of Σ. Then, by Corollary 1, $G = \{\sigma \in S_n \mid (S_n)_i C = (S_n)_i \sigma C \sigma^{-1}, \text{for } i = 1, 2, \ldots, n\}$. Let $A = \{\sigma \in S_n \mid \sigma C \sigma^{-1} = C\}$. Then, $A \leq G$, that is, A is a subgroup of G.

As an application of Theorem 1, we consider the problem of complete coloring of regular digraphs. Let $\Gamma = Cos(S_n, (S_n)_1, D)$ be a connected regular digraph. According to [5], $C \subseteq S_n$ is a complete coloring of Γ if $\langle C \rangle = S_n$ and $\Gamma \cong \Sigma$, where $\Sigma = Cos(S_n, (S_n)_1, C)$ is a connected regular digraph. Thus $CC^{-1} \subseteq 1 \cup S$ and $\Sigma \cong Cos(S_n, (S_n)_1, C^{\sigma})$ for $\sigma \in S_n$. By Theorem 1, we know that $C \subseteq S_n$ is a complete coloring of Γ iff $(S_n)_i D = (S_n)_i C$, $i = 1, 2, \ldots, n$, $CC^{-1} \subseteq 1 \cup S$, and $\langle C \rangle = S_n$. Let $T = \bigcap_{i-1}^{n} (S_n)_i D$. If $C \subseteq T$, $CC^{-1} \subseteq 1 \cup S$, and $|C| = |D|$, then we have $(S_n)_i D = (S_n)_i C$. Therefore, we have arrived at the following result:

Theorem 2. *The set $C \subseteq S_n$ is a complete coloring of Γ iff $C \subseteq T$, $CC^{-1} \subseteq 1 \cup S$, $|C| = |D|$, and $\langle C \rangle = S_n$.*

Fig. 1 is a complete digraph K_4^+ with loops, whose coset graph representation is $Cos(P, P_1, D)$, with $P = \langle(1234)\rangle\,\langle(13)\rangle$, $P_1 = \langle(24)\rangle$, $D = \{(1234), (1432), (13), (24)\}$. The graph $Cos(S_4, (S_4)_1, C)$ in Fig. 2 is a complete coloring of the one in Fig. 1, where $C = \{(1234), (132), (143), (24)\}$.

Remark 1. Finding an efficient algorithm for determining whether a regular digraph possesses a complete coloring is an open problem worthy of further research.

Let us now consider another application of Theorem 1 to the problem of normality of Cayley digraphs. Let P be a finite group and $C \subseteq P$. The Cayley digraph $Cay(P, C)$ of P with respect to C has the vertex set P and the arc set $\{(g, gc) \mid g \in P, c \in C\}$. Now, let $X = Cay(P, C)$, $G = Aut(X)$, and $L(P)$ be the left regular transformation group. The Cayley graph X is called normal if $L(P)$ is a normal subgroup of G. Letting $L = L(P)$, we have $X \cong Cos(L, L_1, C)$, the coset graph of L with respect to L_1 and C, where 1 is the identity of P. We have $L_1 = e$, the identity of L. By Theorem 1, we have $G = \{\sigma \in S_n \mid (S_n)_i C = (S_n)_i \sigma C \sigma^{-1}$ for $i = 1, 2, \ldots, n\}$, where $n = |P|$. Let $Aut(P, C) = \{\alpha \in Aut(P) \mid C^{\alpha} = C\}$. We can now establish the result of Theorem 3 directly from the following proposition:

Proposition 1. *If* $P = \langle C \rangle$, *then we have* $A = L(P)Aut(P, C)$, *where* $A = \{\sigma \in S_n \mid \sigma C \sigma^{-1} = C\}$.

Proof. We first note that, according to [18], $N_G(L(P)) = L(P)Aut(P, C)$. Since $P = \langle C \rangle$, we have $A \leq N_G(L(P))$. On the other hand, we also have $L(P)Aut(P, C) \leq A$. Thus, $A = L(P)Aut(P, C)$. □

Theorem 3. *If* $P = \langle C \rangle$, *then* X *is normal iff* $G = A$.

We conclude this section with an example.

Example 1. The complete graph K_4 is normal as a Cayley graph of the group $Z_2 \times Z_2$. Let $C = \{(12)(34), (13)(24), (14)(23)\}$, $P = \langle C \rangle$. Then, $P \cong Z_2 \times Z_2$, $X = Cay(P, C) = K_4$, $G = Aut(X) = S_4$. Let $\alpha \in G$, then $\alpha C \alpha^{-1} = C$ and, thus, $\alpha \in A = \{\sigma \in S_4 \mid \sigma C \sigma^{-1} = C\}$. Hence, $X = K_4$ is normal as the graph $Cay(P, C)$.

5 Conclusion

This study was motivated by a class of low-diameter interconnection networks presented by Melhem [10]. We have shown that LDI networks are identical to the previously known generalized de Bruijn graphs, exposing and correcting an error in the proposed decomposition of such networks into permutations. In addition, we have shown that the corrected decomposition is an instance of arc coloring of coset graphs. This led us to the derivation of some general results on arc coloring of coset graphs, whose applications include complete arc colorings and normality of coset graphs.

Finding an efficient algorithm for constructing a complete arc coloring for a coset graph is an interesting open problem. We believe that our results will find additional applications in network design and evaluation and plan to pursue a number of problems in this regard.

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