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On isomorphisms and similarities between generalized Petersen networks and periodically regular chordal rings

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Abstract

Generalized Petersen (GP) networks and periodically regular chordal (PRC) rings have been proposed independently to ameliorate the high latency and extreme fragility of simple ring networks. In this paper, we note that certain GP networks are isomorphic to suitably constructed PRC rings, while other varieties correspond to PRC rings that closely approximate their topological and performance attributes. In the absence of equivalence and similarity proofs in the opposite direction, our results indicate that PRC rings may be preferable to GP networks in the sense of covering a broader family of networks and offering greater flexibility in cost-performance tradeoffs.

Keywords: Chordal ring; Double-ring network; Hamiltonicity; Interconnection networks; Loop; Parallel processing; Petersen graph; Ring; Symmetric network

1. Introduction

Numerous topologies for interconnecting processing nodes or full-fledged computers have been proposed by researchers in parallel computing and communication networks [3,8]. It is often difficult to compare such networks with respect to their suitability for a particular application domain [7]. Such a comparison entails a multitude of modeling and analyses efforts to assess the networks' static attributes (diameter, average distance, bisection, VLSI layout area) and dynamic properties (routing algorithms, deadlock prevention, traffic balance, fault tolerance). Thus, it is counterproductive to introduce new classes of networks unless they can be shown to offer tangible advantages in one or more of the aforementioned areas for a reasonably wide application domain.

Popularity of the ring topology stems from its structural simplicity and efficient routing protocols. These properties lead to low implementation cost and high communication throughput, with long delays and extreme fragility being potential drawbacks. One way to ameliorate these drawbacks is to endow a p-node ring with skip links or chords, forming a chordal ring $CR(p; s_1, s_2, \ldots, s_g)$, where the skip distances or chord lengths s_i imply that a node j is linked to nodes $j \pm s_i$, besides the ring links connecting it to $j \pm 1$. Another way is to introduce a secondary ring to form a double-ring network (Fig. 1(a)). Combining the two ideas of

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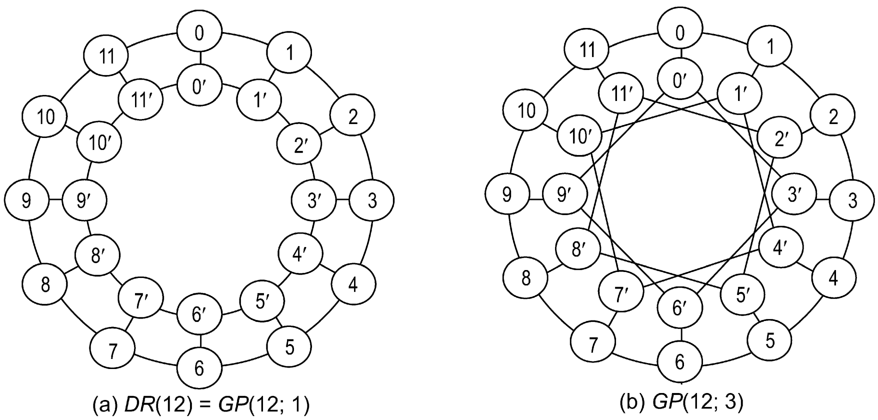

Fig. 1. Examples of double-ring (DR) and generalized Petersen (GP) networks.

secondary ring and chordal connections leads to the generalized Petersen graph (Fig. 1(b)), where the inner "ring" has $\pm s$ skip links instead of ± 1 ring links [4]. A special case of $GP(n; s)$, the generalized Petersen graph with $p = 2n$ nodes and skip distance s, is the generalized double-ring (GDR) network which restricts n and s to be relatively prime [11], thus ensuring that the inner part is in fact a ring.

Our goal in this paper is to show that any GP, and thus GDR, network corresponds to a PRC ring that is either isomorphic to it or else very closely approximates its key architectural parameters. Put another way, PRC rings are more versatile than GP networks. Thus, we advocate the use of PRC rings where GP networks have previously been applied. Any network class that can replace, or show benefits over, multiple distinct classes of networks is useful in the repertoire of parallel computer architect. For example, it allows single streams of effort for algorithm development, provision of fault tolerance, analysis of energy efficiency, and packaging considerations to replace multiple current streams.

2. Background and definitions

We label the n nodes of a ring from 0 to $n-1$. Node-index expressions are taken to be modulo n, with node $n-1$ being adjacent to node 0. For a DR network, with $p = 2n$ nodes, the outer nodes 0 to $n-1$ and inner nodes $0'$ to $(n-1)'$ are connected by outer edges $(j, j+1)$, inner edges $(j', (j+1)')$, and spoke edges (j, j'). For GP or GDR, the inner edges become $(j', (j+s)')$. Among known results [1] for $GP(n; s)$ is the fact that it is almost always Hamiltonian (i.e., contains or embeds a $2n$-node cycle), the only exceptions occurring for $GP(n; 2)$ when $n = 5 \bmod 6$ (see Theorem 13.1 on p. 316 of [5]). Besides the Petersen graph $P \equiv GP(5; 2)$, other noteworthy special cases of $GP(n; s)$ are the cubical graph $GP(4; 1)$, the Möbius–Kantor graph $GP(8; 3)$, the Desargues graph $GP(10; 3)$, the symmetric cubic graph $F_{24} \equiv GP(12; 5)$, and the prism graph $GP(n; 1)$.

The double-ring network of Fig. 1(a), also known as the prism graph, was first proposed to improve the robustness of a simple ring, which is particularly vulnerable to cable-cut accidents when deployed in underground trenches. Pedersen and coworkers [6,11,12] studied generalized double-ring networks (N2R networks, in their terminology) in comparison with simple DR networks and degree-3 chordal rings, taking topological parameters, such as diameter and average internode distance, as well as reliability into consideration. While their simulation-based results do not advance our theoretical understanding GDR networks, they do point to potential advantages of $GDR(n; s)$ in comparison with other networks of similar cost and complexity.

We next define the notion of periodic regularity in chordal rings [9]. Consider a p-node chordal ring $CR(p; s_1, s_2, \ldots, s_g)$, with g skip links of lengths $s_1 < s_2 < \cdots < s_g$. To reduce the node degree of such a network from $2g + 2$ to 4, while maintaining some of its desirable attributes such as small diameter and average internode distance, we may pursue the following strategy. Let g divide p. Rather than provide all g different skip links for each node, we divide the nodes into g classes according to the residue of their indices modulo g. We assign a single skip link type to each class of nodes. As shown in Fig. 2, node j may be provided with two skip links to nodes $j \pm s_{g-(j \bmod g)}$. Both the diameter and average internode distance increase by no more than g as a result of this strategy, which constitutes a form of pruning. Given that g is relatively small in practice, the loss of performance due to the increased

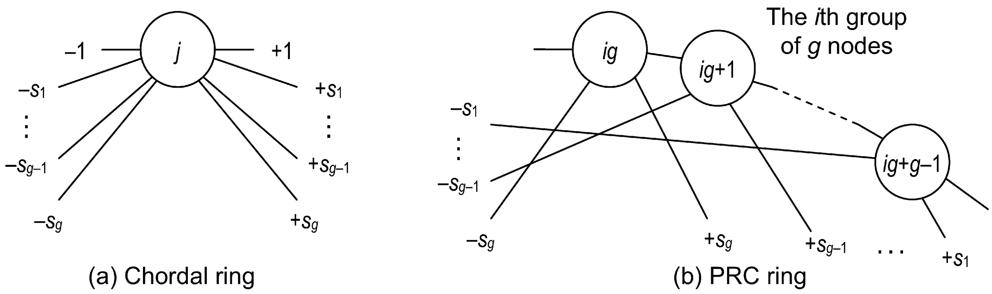

Fig. 2. Node structure in chordal rings and PRC rings, showing degree reduction from $2g + 2$ to 4.

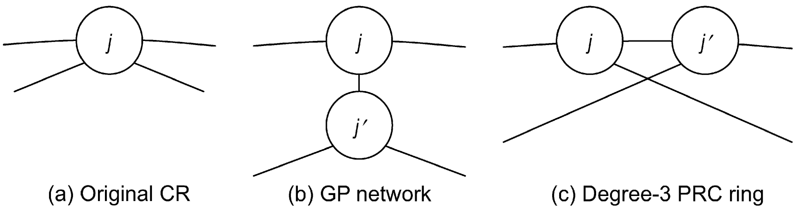

Fig. 3. Two ways of replacing a degree-4 node with two degree-3 nodes.

routing distances is negligible (aggregate network bandwidth is a different story, of course). Certain conditions on the skip distances are required if the resulting network is to be regular with regard to node degree. However, here we are not interested in PRC rings in their full generality, so there is no need to discuss such conditions.

The construction leading to degree-4 PRC rings (Fig. 2) is a special case of a node degree reduction strategy in which a node of degree d is replaced by a network of lower-degree nodes that collectively provide d external links to serve as the original links of the replaced node. Fig. 3 shows how $GP(n; s)$ and degree-3 PRC networks, $PRC_3(2n; s)$, represent two different ways of applying this strategy to reduce the node complexity of a degree-4 chordal ring: a GP network is obtained when each CR node is replaced by a 2-node network in the radial direction, whereas a PRC_3 ring network is obtained when substitution is in the tangential direction. A natural question that arises is how $2n$-node, degree-3 GP networks and PRC_3 rings compare with regard to static parameters and dynamic performance attributes. We endeavor to show that PRC_3 rings are preferable to GP networks.

3. GP networks as PRC rings

DR networks are known to be Hamiltonian. A spanning cycle of $DR(n)$ is readily traced by using any two adjacent spoke edges, plus all but one of the outer and inner edges. A different Hamiltonian cycle can be formed that moves between outer and inner rings, taking one step in each before switching to the other. This leads to an alternate drawing of the network of Fig. 1(a), shown in Fig. 4(a). The PRC ring isomorphic to $DR(12)$ has a group size of $g = 2$ and skip distances of $+3$ for even nodes and -3 for odd nodes. Clearly, one can use this construction for any $DR(n)$ with n even. It remains to establish whether an isomorphic PRC ring can be found for odd n. The following theorem shows the answer to be negative.

Theorem 1. *The double-ring network $DR(n)$ is a PRC ring if and only if n is even.*

Proof. The "if" part is straightforward, given the preceding construction. To prove the "only if" part, let n be odd. Clearly, we cannot trace a Hamiltonian cycle by alternating between the outer and inner rings, switching on every other step. A Hamiltonian cycle must thus take a hop somewhere from the outer/inner to the inner/outer loop, remaining there for at least 2 hops. Without loss of generality, let the cycle include the segment $0' \to 0 \to 1 \to 2$. It is easy to see that the postulated Hamiltonian cycle cannot be completed, except if the inner and outer parts of the path continue separately, eventually rejoining via the spoke edge $((n-1), (n-1)')$. Thus, only two spoke links are part of the Hamiltonian path. However, spoke links are not interchangeable with the outer or inner links (the network

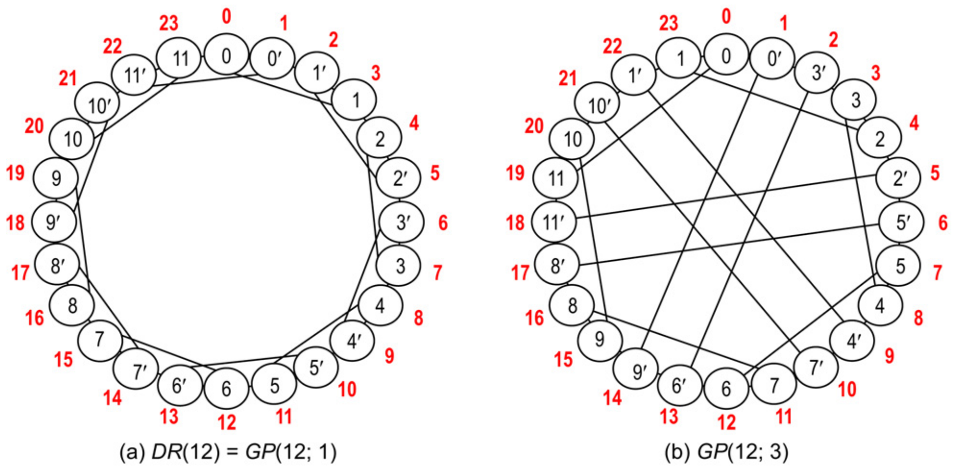

Fig. 4. PRC ring views of $DR(12)$ and $GP(12; 3)$.

is not edge-transitive). To see this, consider that each spoke link is part of two different cycles of length 4, whereas each outer/inner link belongs to only one such cycle. Having only two spoke links in the Hamiltonian cycle forces a period of at least $n/2$. Note that in a PRC ring, any two skip links of the same type s_i that are separated by g nodes (i.e., they emanate from nodes j and $j+g$, where g is the period) are completely interchangeable. □

For GP networks, the construction is more complicated. Consider $GP(12; 3)$ in Fig. 1(b). To trace a Hamiltonian cycle for this network in a manner as to produce an isomorphic PRC ring, it is necessary, but not sufficient, to utilize spoke links at equal intervals. The condition is a necessary because periodicity (with period g) dictates that links emanating from nodes j and $j + hg$ be interchangeable. One possibility is shown in Fig. 4(b), which is a PRC ring with $g = 4$ and skip distances $-5, -11, +11, +5$.

Theorem 2. *$GP(n; s)$ is a PRC ring if n is even, $s > 1$ is odd, and $n = 2 \bmod (s-1)$.*

Proof. Note that the condition "n even" is redundant, because it is implied by "s odd" and $n = 2 \bmod (s-1)$. Begin a Hamiltonian cycle at node 0. Proceed as follows, taking spoke links on alternate steps:

$$0\ \underline{0'}\ s'\ s\ (s-1)\ \underline{(s-1)'}\ (2s-1)'\ (2s-1)\ (2s-2)\ \underline{(2s-2)'}\ \ldots$$

The pattern in the partial sequence above is periodic, with a period of 4, as evidenced by the underlined inner-ring node numbers that are multiples of $s-1$. If continuation of the underlined sequence of node numbers leads to the node $(s-2)'$, the pattern can be repeated by proceeding to $(s-2)$, $(s-3)$, and $(s-3)'$. From there, each node number is shifted by -2 relative to those in the previous repetition, eventually closing the path via the final nodes $1'$ and 1. Thus, because of the wraparound when going beyond the starting nodes 0 and $0'$, some multiple of $s-1$ must equal $n+s-3$, that is, $l(s-1) = n+s-3$. To make the right-hand side of this equation a multiple of $s-1$, we must have $n = 2 \bmod (s-1)$. In the special case of $s = 3$, the latter condition becomes $n = 2 \bmod 2$, which is the same as "n even". □

Theorem 2 provides sufficient conditions for $GP(n; s)$ to be a PRC ring. If the postulated conditions are satisfied, then the isomorphic PRC ring has a period of 4. At this point, we do not know whether these conditions are necessary as well, although we strongly suspect that they are. Note that the final condition on n in the statement of Theorem 2, that is, $n = 2 \bmod (s-1)$, is not as restrictive as it appears at first glance. For example, it is always satisfied for $s = 3$, it is satisfied for $s = 5$ provided that n is not a multiple of 4, and, more generally, it holds for one out of every $(s-1)/2$ consecutive even values of n. Therefore, a PRC ring network of size close to that of an arbitrary DR or GP network can always be constructed. This notion is formalized in the following theorem.

Theorem 3. *Given the diameter-D network $GP(n; s)$ with n even, s odd $(s \geqslant 5)$, and $n \neq 2 \bmod (s-1)$, we can construct a PRC ring with no more than $2(n+s-3)$ nodes whose diameter is at most $D+1$.*

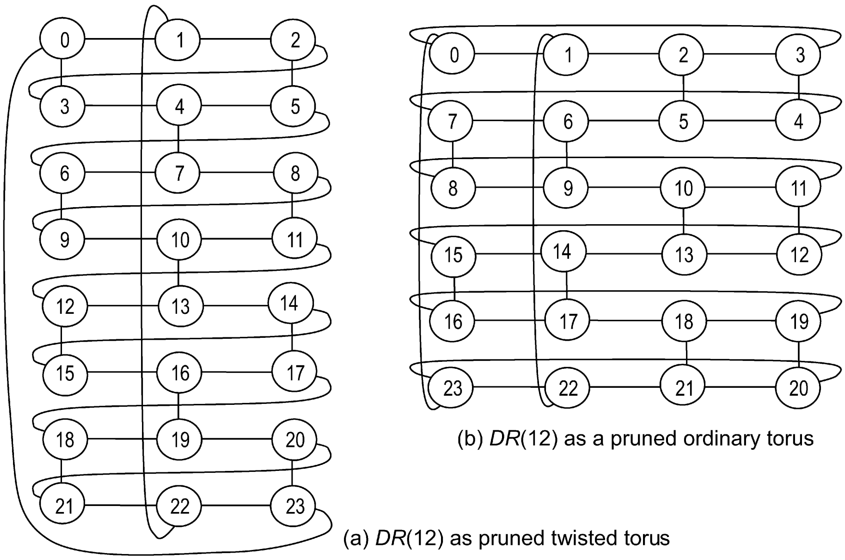

Fig. 5. A PRC ring network and its representations as a pruned (twisted) torus.

Proof. Let $n = m \bmod (s-1)$, where m is even and $m \neq 2$. The postulated PRC ring is derived from the Hamiltonian-cycle construction of Theorem 2, after adding $2[(s+1-m) \bmod (s-1)]$ nodes to $GP(n; s)$ to transform it into $GP(n + (s+1-m) \bmod (s-1); s)$. Because $(s+1) = 2 \bmod (s-1)$, it is readily seen that the number of nodes in the expanded network is $2 \bmod (s-1)$, thus making the Hamiltonian cycle construction method of Theorem 2 applicable. The maximum enlargement of n during the transformation above is $s-3$ (which occurs for $m = 4$), leading to the maximum of $2(n+s-3)$ nodes in the network. The fact that the diameter of the expanded network is no more than $D+1$ is readily provable by observing that there can be no more than an extra skip link on the shortest path from any source node to a destination node, compared with a corresponding diametral path in $GP(n; s)$. □

4. Some practical implications

Based on Theorem 3, even when a GP network is not isomorphic to a PRC ring, we can use a PRC ring of comparable size and diameter in lieu of the former. The relative size increase in this replacement is no greater than $(s-3)/n$. In general, the optimal value of s, that is, the skip distance that minimizes the network diameter and its average distance, is $s^{\text{opt}} = \Theta(n^{1/2})$. The worst-case relative size increase from GP network to the derived PRC ring is thus $O(n^{-1/2})$. This is no worse than the effect of adding a single row or column to a 2D square mesh or torus network. The effect on diameter is also comparable.

If PRC_3 rings are to be used in parallel architectures, either as direct choices or as substitutes for GP networks, their structural properties must be contrasted with other fixed-degree networks of comparable costs. We have shown PRC rings to be isomorphic or topologically similar to GP networks. Thus, results of prior research showing GDR networks (special cases of GP networks) to have advantages over other degree-3 networks carry over to PRC rings. As depicted in Fig. 5(a), a p-node PRC_3 ring is isomorphic to an $s \times (p/s)$ pruned twisted torus, where s is the skip distance. Fig. 5(b) shows that some PRC rings are isomorphic to ordinary pruned tori. Thus, documented advantages of pruned tori [10] carry over to PRC rings, as do comparative evaluations between pruned tori and other fixed-degree networks.

The next issue to be considered is that of incremental scalability, that is, the extent to which a network can be expanded to include a larger number of nodes. In this respect, hypercube networks are not very scalable for two different reasons: the fairly large step size in going from one configuration to the next possible configuration (with twice as many nodes), and the increase in node degree to allow this expansion. Similarly, the incremental scalability of many fixed-degree derivatives of hypercube networks suffers from the requirement that the network size be a power of 2. Ring and mesh networks, on the other hand, are more readily scalable. The incremental scalability of PRC rings follows from the pruned twisted torus equivalence depicted in Fig. 5.

5. Conclusion

We have shown that many (generalized) double-ring networks and generalized Petersen graphs are isomorphic to periodically regular chordal rings. Additionally, we have demonstrated that when a $GP(n; s)$ network is not isomorphic to a PRC ring, it can be replaced by a PRC ring that is at most only slightly larger and has a diameter that is either the same or one unit more. Furthermore, the replacement PRC ring enjoys identical fault tolerance and robustness parameters, including 3-connectivity [13]. The fairly complicated routing algorithms of GP networks [2] become simpler with the PRC ring view. It is certainly an advantage for a number of routing functions to have a Hamiltonian cycle readily visible and not requiring any calculations to derive.

In the absence of equivalence and similarity proofs in the opposite direction, our results indicate that PRC rings may be preferable to GP networks in the sense of covering a broader family of networks and offering greater flexibility in cost-performance tradeoffs. Exploring conditions and application contexts under which performance and fault tolerance benefits could materialize constitutes a possible direction for further research. Other areas for future research include the derivation of exact values (closed-form formulas) for the diameter, bisection width, and fault diameter of PRC_3 rings. The node symmetry of PRC rings and their periodic regularity (the next best thing, in the absence of complete edge-symmetry) make both analytical and experimental evaluations feasible with moderate effort.

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