Number Representation and Arithmetic in the Human Brain

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***Abstract*—The human brain as a platform for number representation and arithmetic is a complex system that involves a large bilateral network spanning multiple aspects of cognition. Numbers are encoded in the so-called "triple code" that entails verbal, quantitative, and written forms. A healthy individual's brain typically activates these regions in various capacities when performing calculations with multiplication vs. addition, exact computation vs. approximation, and large vs. small operands. In comparison to artificial systems, a human brain is likely to rely more on memorization than counting or sequential arithmetic. This review is motivated by the fact that all the attributes just cited hold potentially valuable lessons for computer engineers aiming for compact, efficient, and energy-frugal system design.**

Keywords—Biological computation, Biological number sense, Brain-inspired computing, Digital-analog circuits, Enumeration, Mental arithmetic, Neuronal number code, Numerosity

I. Introduction

Numeracy is generally considered a basic capacity of the human brain [1]. Our brain has dedicated circuitry for recognizing the number of objects in a set and for providing us with the intuition needed to acquire formal arithmetic [2]. As computer engineering researchers, we are motivated to understand how the brain represents and processes numbers, in case there are lessons to learn and methods to scrounge in building compact, algorithmically-efficient, and energy-frugal computing systems. Neurons are known to be much slower, but orders of magnitude more energy-efficient, than state-of-the-art digital circuits. We also know that lower-speed digital logic consumes less energy. So, there may indeed be lessons for us in the brain's computational scheme, a key motivating factor for the emerging field of neuromorphic computing [3].

Human brain activity when it is engaged in arithmetic tasks is directly linked to the general computational abilities of a subject [4]. So, an understanding of such activities can lead not only to smarter, more capable computers but also to practical strategies for improving the computational performance of humans or human-machine combinations. It isn't out of the question that a flow of methods in the reverse direction can be established to improve the performance of the human brain based on computational strategies borrowed from digital and analog electronics.

A first order of business is to consider what has been called "number sense," that is, a basic ability—one that is innate and not developed by mathematical training—to conceptualize and manipulate numerical quantities [5], [6]. Therefore, we expect number sense to be a very common quality in neural development. If humans have some genetic predisposition for mathematical ability, then other animals should have some traces of it as well [7] [8] [9], and, surely enough, the presence of basic arithmetic ability and number sense has been verified experimentally in many other animals, including pigeons, rats, dolphins, and, of course, non-human primates [5].

A good review of experiments designed to chronologically map the development of mathematical abilities in preverbal infants is given by Dehaene [5]. The studies discussed in this collection present auditory and visual stimuli to infants, reporting variations in attention by monitoring changes in the directions in which they look and the times when they look in a given direction. In one experiment that tested infants' evaluation of mathematical statements like $1 + 1 = 2$, for example, single objects were presented in turn and moved out of sight, leading to infants expressing surprise when a different number of objects was retrieved [5] [6] [10]. Not only do these kinds of experiments indicate that infants have a degree of number sense similar to that found in other animals, but they also imply that number sense may have a role in object permanence.

Number sense is what allows us to understand approximations, perform numerical comparisons (few vs. many), and count small discrete values. In animals, this counting ability is believed to be favorable to evolution, because it may allow animals to track the number of predators around, as well as help animals in groups determine whether an enemy group is small enough to conquer [6] [7]. Overall, elementary mathematical abilities innate to other animals are similar to human abilities at the infant stages. Because of this correspondence, we focus on number representation and arithmetic in the human brain.

Much research has been done on the representation of numbers and performance of arithmetic operations in the human brain. In parallel, novel number representation schemes constitute an active area of research in digital computer arithmetic [11]. In this paper, our focus is on explaining key elements whose understanding opens up the subject matter, provides the needed nomenclature for discussion, and allows us to pursue more details, as needed. These elements are discussed in Sections II-VIII of the paper, with conclusions, including brief mentions of the current limits of our understanding along with open problems, presented in Section IX.

II. Sense of Small Numbers

Looking at two apples on one side and three oranges on the other, we immediately recognize the different cardinalities of the two sets, without thinking. The effortless recognition of small numbers in an exact way extends perhaps to half-dozen, and to a couple-dozen with less accuracy. This sense is what enables a grocery-store clerk to quickly determine whether the number of items in a shopping cart might exceed the limit for express checkout, without actually counting. Numeric abilities of individuals may differ based on experience and training, but the pertinent parameters are always in the same ballpark.

This instinctive ability originates from dedicated circuits in the brain. A small area deals with numerosity, that is, determination of a set's size, along a continuous map, much like the maps for more fundamental or basic senses [12]. Abstract, higher-level cognitive functions typically do not have built-in maps, so numerosity is a baser, intuitive sense. It is believed that one side of this mapped region is devoted to small numbers and an adjacent region to larger numbers, with an increasingly sparser mapping to this region as the number value grows.

Recognition of small numbers is accomplished by single-neuron sites in the prefrontal cortex and a couple of other regions associated with mathematical processing [13]. So, if a subject is shown three dots on a single sheet of paper, or three sheets of paper, each bearing a single dot, in rapid succession, neurons associated with the concept of "3" light up. These specialized neurons will fire strongly when their specific number is encountered; they exhibit weaker firings for nearby numbers (Fig. 1). So, the sense of small numbers in the brain, while exact as far as perception is concerned, has a kind of fuzziness [14] at the neural-activity and signaling level.

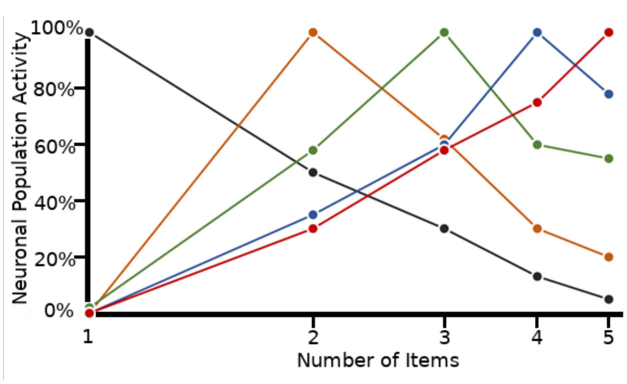

Fig. 1. Neural firing activity for small numbers.

The existence of words for small numbers in ancient languages that were in use long before formal mathematics was developed confirms that small numbers have innate representations in our brain; how linguistics helps with the study of various branches of science is a fascinating story! The hypothesis that only small numbers have direct brain circuits is strengthened by the fact that several known ancient languages had words only for very small numbers [15] or no specific number words at all, just words for "a few" and "some"—the so-called "anumeric languages" [16]

III. Large Numbers and Scale

As mentioned in Section II, the brain map region enabling numerosity devotes less area to numbers as they grow larger. This is in line with the idea that without mathematical training the brain uses a logarithmic scale for numbers [9] [10] [17], known as the Weber-Fechner Law [18]. Logarithmic number representation in the brain results in improved speed and precision with smaller numbers and, more generally, it is naturally efficient and robust, leading to a broad range of applications in computing [19], besides its use in the brain.

Even though our sense of numbers is intrinsically logarithmic, over time, we develop the skill of visualizing the number line in linear form. Both logarithmic and linear visualizations help in comparing numbers, getting a sense of their scales, and doing mental arithmetic on them [6] [20]. With the logarithmic scale, each doubling in the number's value moves us a fixed distance to the right on the number line, thus helping in the visualization of relative magnitudes. The linear scale is more useful in visualizing absolute differences (Fig. 2).

In addition to this physical representation, it is theorized that humans internally encode numerical quantities alongside their Arabic digit numerals and their verbal equivalents. The three types of encoding make up what is known as the triple-code [7] and contribute to highly-interconnected and partially-robust numerical profiles in the human brain.

The locality of cortical activity varies for different operations (addition versus multiplication), small (~ 1-5) or large (typically > 5) quantities, and exact or approximate calculations. Exact calculations are those whose output requires full evaluation to verify correctness. Usually this type of calculation is performed to gather evidence of the size effect. For example, a statement such as 19 + 18 ?= 37 or ?= 39 might be presented, and a participating individual asked to choose the correct answer. On the other hand, approximate calculations are defined to be computations whose answers are chosen based on their closeness to the approximate real answer. For instance, if 19 + 18 ?= 250 is presented, the participant easily rejects it as false, without evaluating the exact answer, given the large distance between the operands and result.

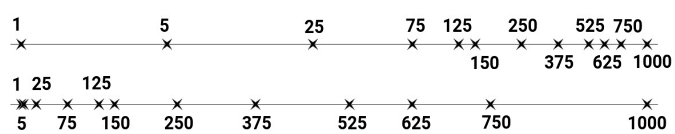

Fig. 2. Logarithmic and linear number lines.

That the brain takes more time to verify exact calculations whose supplied potential answers have shorter distance between them, like 37 and 39, is well-established. The phenomenon is fittingly called the distance effect, and the tests used in exact and approximate computation are appropriate for observing it across addition, subtraction, and multiplication operations. Another limitation to computation speed is known as "the size effect," which suggests that arithmetic or comparisons with larger operands are more prone to errors [5].

Some experiments have been performed to observe whether additional approximate strategies, such as the odd-even rule in addition, can be used to speed up calculations [21]. But we must look at neural activation patterns and the organization of numbers in the brain to better understand the methods employed during computation.

IV. Brain's Arithmetic Regions

As evidenced by studies of the distance effect, our brain distinguishes large numerical differences quite easily, such as that between 50 and 500 represented as sets of dots. However, this innate ability must be augmented with symbolic representation and processing abilities in order to distinguish, say, 50 from 55.

To understand the mathematical abilities in humans, we can disregard much of the brain activity dedicated to other autonomous and sensory processing tasks, including activity in the pituitary gland, auditory lobes, and so on. Still, quite a few brain areas call for closer study, because arithmetic and number processing entail a highly-integrated neural network within the human brain (Fig. 3). In fact, this network extends across cerebral areas used in language processing and spatial reasoning, among others [22].

Activation of neural subnetworks and neighboring cortical areas depends primarily on the calculation type, whether it's exact or approximate, as well as the operation, and the size of and relative distance between the operands. The extent of integration of mathematical ability in the brain is typified by the observation that, upon the presentation of numbers, other arithmetic facts like the sum and product are autonomously activated—a process known as obligatory activation [23]. This activation suggests that certain calculations, like multiplication tables, are memorized [6] [7] [24].

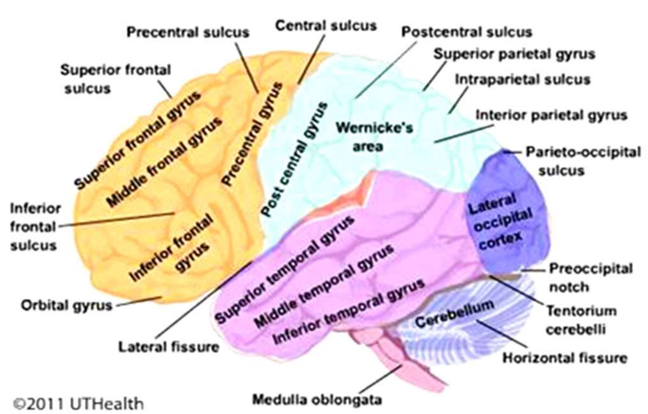

Fig. 3. Brain areas linked to numbers and arithmetic [25].

The memorization of arithmetic facts is tightly integrated with language processing—that is, through numerical storage in verbal and written or symbolic memory locations. Numbers are likely encoded in these locations in the language in which they were first learned. From this perspective, it is the case that bilinguals will perform computations slower if the numbers are presented in their non-primary language [6] [24] [26]. However, the triple-code encoding abstraction of the connection between mathematical facts and language in the brain is superficial. On a deeper level, the storage of numerical facts is chosen closely to language due to the proximity of the cortical areas activated in arithmetic to Wernicke's area, the region responsible for language comprehension.

Near Wernicke's area is the intraparietal sulcus (Fig. 3), likely responsible for mathematical processing [6] [7] [18] [22]. Broca's area, responsible for speech formation, is nearby. However, as mentioned previously, mathematical processing is not limited to one region of the brain. In both addition and multiplication operations, activation of the intraparietal sulcus is commonly observed [6] [7] [18], but other related areas also activate depending on the size of the operands and precision of the calculation.

In general, the parietal lobe activates during approximate calculations or those with large operands, while the prefrontal cortex activates during exact calculations [5] [10] [18] [22]. In calculations with larger numbers, or more complicated arithmetic demanding exact results, both regions show greater activation [22]. The entire mathematical network that forms the basis for the triple-code theory (that numbers are stored in quantitative, verbal, and symbolic forms) are spread out and overlap other areas involved in cognitive activity [27]. This includes the area responsible for verbal numerical representation, which is close to Wernicke's area and Broca's area [5] [22]. Both areas are used in language development and complicated verbal computational tasks, and are known to activate for word-based arithmetic problems.

V. Brain's Computational Architecture

Despite their extensive connections, the networks responsible for numeracy are not symmetric in the human brain with respect to cerebral hemispheres. Networks responsible for magnitude representation and number comparison are located in the prefrontal cortex and are common to both hemispheres. Visual encoding of numbers is stored in the angular gyrus, near the parietal and occipital lobes, and is also common to both hemispheres. Each of these areas communicates to each of the other areas and also engages in bilateral hemispheric activity. However, the verbal encoding and storage of arithmetic facts is only unilateral, even though it communicates with the other areas [5] [22]. Much detail provided in the cited references has been omitted here for brevity.

Equipped with the triple code, we can now use our working knowledge of the human brain's interconnectedness to understand the brain as an architecture. Encoding numbers and arithmetic facts in three separate but coupled formats in a noisy biological environment is somewhat prone to calculation-specific errors. We will now consider the types of errors incurred during mathematical computation, and discuss how they may manifest as an artefact of the architecture.

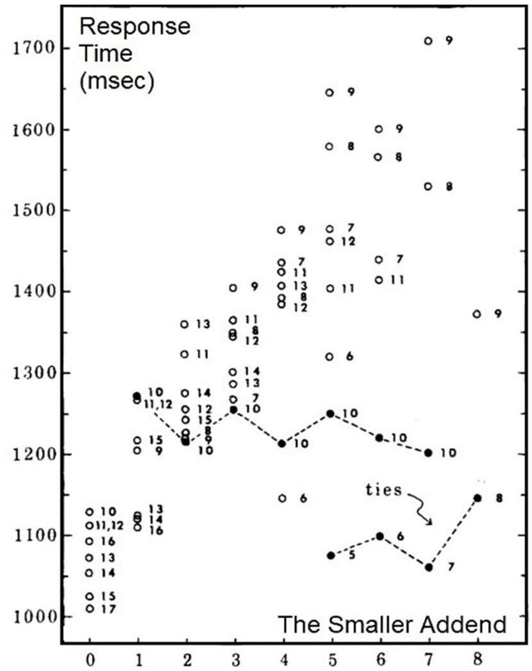

Fig. 4. Mental arithmetic for single-digit addition [21].

It has been shown that quantity magnitude errors are the dominant source of errors during nonverbal quantity estimation, and not errors due to relative difference between quantities [26]. This, in effect, states that the size effect is more likely to incur computational error over the distance effect. There is evidence that latency of and errors in exact verbal calculation also increase in the case of some multiplications whose results are considered "close" to each other [5] [20] [24]. As an example, take the statements 7 × 8 = 56 and 7 × 8 = 63. Though the result of the second is incorrect, the multiplicands that would yield the result 63 are close to the supplied multiplicands, and thus the rejection of the second statement may be slower or the process more prone to error. In a way, we might supplement this example by providing an analogy to electronics, and state that the small cerebral locality in which the multiplication table is stored, along with the size effect between numbers 7, 8, and 9, contributes to a highly noisy readout [26].

Interestingly, delays in solving arithmetic in the human brain tend to increase when a carry operation is involved in the case of addition of two numbers [21]. As shown in Fig. 4, which depicts the addition latency with different addends, the latency is greatest when one addend is 9, which is guaranteed to generate a carry if the second addend is nonzero.

VI. Brain vs. Neural Networks

Though the errors mentioned in Section V seem costly to any digital system, they can often be circumvented to some degree with additional mathematical training. In healthy individuals, we may consider arithmetic ability to be consistent on average, disregarding exceptional training in select people [5]. Calculating ability varies by age, as well. For example, infants tend to perform addition using counting methods, while adults tend to rely on memorization [20] [24]. In analogy to conventional logic, counting maybe viewed as an add-one methodology and memory access as table lookup.

Artificial systems (artificial neurons, in particular) perform arithmetic quite differently [28]. Neural networks still often rely on logic schemes with a fairly constrained set of inputs compared to a biological neuron with many more dendritic synapses. But the most important difference is that artificial systems are a projection of analog computation onto a level- or spike-based voltage domain. This on its own implies an extreme reduction in dimensionality, given that the chemical-based information exchanges in the neuron are lost; however there have been significant efforts to create brain-like systems with high connection density and throughput in CMOS.

Pioneering neuromorphic-computing projects include IBM's TrueNorth, Stanford's Neurogrid, BrainScaleS, and SpiNNaker, which use techniques ranging from real-time modeling to object recognition and robotic control [3]. They are all VLSI-based and, by conventional definition of neuromorphic computing, use some electronic representation of the biological neuron as the fundamental circuit element. At least for the neuromorphic computing examples referenced here, the theme of neuromorphic computers is akin to distributed computing with high inter-module connectivity. This essentially mimics, in two dimensions, the brain's three-dimensional construction.

Limitations of artificial circuits include power delivery and cooling, as well as multi-layer scaling. Resolving these issues, perhaps through sub-threshold transistors, may open the way to further progress. While modern FPGAs allow reasonably capable implementations [29], the eventual solution may entail something unexpected, like a chemical medium, with artificial neurons encased in power delivery "membranes."

In artificial perceptrons, inputs are weighted and tallied, and the perceptron fires when the tally exceeds its built-in threshold. Adders can be built from threshold networks with suitable connections and weights [30]. Though closer to the biological version than the CMOS-logic realization, such an adder still falls short. For instance, if addition is most-likely performed by memorization, then the choice to use sequential circuits is totally wrong if we want to mimic the human brain.

Up to now, we have remained at the architectural level in discussing arithmetic in the human brain. Let's take a brief look at a signal-based analysis, leaving a more detailed investigation to future work. If we assume a rate-encoding-like communication scheme as in the firing of sensory variables, we can take on a different logic scheme that appears to be more realistic in emulating brain-like computation [31] [32]. Arithmetic can transform pulse streams by excitation, which speeds up the firing rate, or by inhibition, which slows the rate. One potential drawback to high-speed firing rate in CMOS technology, however, is an increased failure rate and therefore a need for greater component redundancy.

Performance-wise, neuromorphic computers and artificial computational elements tend to favor speed and low-noise environments over power and efficiency, while biological systems make the reverse trade-off. These key design differences should be considered if an artificial copy of the brain's architecture is the design goal. To better understand how such a copy might operate, however, we must look at the interface between the structure that transmits information and how that information is interpreted.

VII. Analog, Digital, or Hybrid?

While it is common to think of the human brain as an analog computer [33], whether it is more analog or digital is still debated [34]. However, the use of some form of hybrid analog-digital processing in the human brain is hinted at by Fig. 2. From an algorithmic perspective, residue number systems (RNS) have parallels in biological computation. For example, rats have very strong homing instincts that enable them to quickly return to their nests even in the dark by finding the shortest path [35]. It is postulated that their homing instincts may be bolstered by the formation of a spatial grid in which they orient themselves [36]. When using continuous digit-RNS (CD-RNS) as a model for localization in such a grid, transformative operations like addition and multiplication become easy [35]. The dynamic range—or number of uniquely expressible numerical values—becomes a critical point of discussion for maximum grid size and desired errors. It is possible that evolution may have won low error rates in this encoding as well as optimized values to maximize the dynamic range. In this aspect, some similar number system used for spatial orientation is likely to exist in the brains of other species, including humans.

Continuing the discussion of spatial reasoning in humans, consider, as an example, object recognition after 3D rotation. In digital systems, this computation is complicated and involves calculating all possible rotational transformations and performing comparisons; in the human brain this problem becomes less mathematically rigorous and likely dependent instead on the ability to mentally "rotate" the object [33]. In such applications, human brain's computational procedure may be more-aptly characterized as analog.

There is evidence that the brain also exhibits digital behavior. According to Weber's Law, for any stimulus—like ambient temperature or weight carried—the just-noticeable difference (JND), or the minimum detectable change, is a discrete step that is linearly proportional to the reference [34]. In this regard, we can also recognize the brain as being at least partially digital. It is likely that the brain has developed as a synthesis of application-optimized logic choices.

Moreover, in designing brain-like systems, it is important to remember that Boolean arithmetic, whether based on stable voltage levels or transient spikes, is just an abstraction: a digital projection of the brain's analog signaling onto a digital domain. Certain logic schemes, such as race logic, can act as a supplement to spike-based computing [32] and provide useful benchmarks for comparison to biologically-inspired logic domains [37]. Such logic schemes imitate the transient nature of the brain more directly, but spike-based systems may be more error-prone in highly-noisy biological systems like the brain. For this reason, additional processing is needed.

VIII. Robustness and Error-Handling

Given the high noise in biological systems, error detection and correction are critical. In Section III, we discussed the limitations of number sense as an innate ability. Arithmetic with small numbers that fall within the mathematical bounds of this sense are generally error-tolerant, but for numbers outside the bounds, computations—and the potential for errors—become subject to the size effect and distance effect. While the size effect identifies increasing computational error with the growth in operand value, the distance effect describes the growth in error as the quantitative distance between operands shrinks [38].

Importantly, both of these error sources are observed in humans, infants, and other animals, supporting the theory that number sense has been hard-wired in us through evolution.

To some extent, the brain implements its own self-checking measures to counteract computational errors. In multiplication, for example, the brain rejects false mathematical statements ~115 ms faster if the result is not a multiple of either operand (for example, $5 \times 7 = 29$) compared to the case when the result divides one of the operands ($5 \times 7 = 30$). The odd-even rule may additionally help weed out incorrect results more quickly [21]. In addition to some ability for error-checking, the brain has some redundancy, exemplified by its semi-tolerance to physical damage by injury or exceptional conditions.

Patients with cerebral conditions that affect the network described in Section IV still retain some arithmetic ability in quite a few studied cases [5] [22] [39]. Acalculia tends to result from lesions in the intraparietal or angular gyrus regions, where number manipulation is predominantly performed. However, patients with acalculia can still retain verbal number sense, especially with regard to memorization, as in multiplication tables [22]. Similar instances of selective retention of arithmetic ability have been observed in patients with related conditions like dyscalculia, aphasia, and cerebral lesions symptomatic of diseases like Gerstmann syndrome [7] [22] [24].

Architectural redundancy in artificial systems and networks is accomplished in some cases by adding spares. In contrast, the human brain does not have spare lobes to swap in when one becomes defective, even though, in the case of lesions or other conditions that make certain areas of the cerebral network unusable, there are other encoded versions of related data that are still accessible. In the latter case, there is still data and performance loss; in the former, typically no data or performance loss. One might say that the brain's robustness and error-handling features resemble those of distributed computing systems, which use task reassignment and load-balancing to overcome the effects of resource fluctuations.

IX. Conclusion

Our discussion in this paper is intended to serve as a review of existing literature, in an attempt to uncover a holistic view of arithmetic capabilities in the human brain. We discussed number sense and encoding, and the regions of the brain most likely involved in mathematical processing. We also reviewed the size and distance effects and how they alter the processing latency and the likelihood of computational errors.

Computer architects and engineers often think about brain-based computation in terms of artificial neural networks. However, such artificial systems do not yet capture the complexity and efficiency of the human brain. Improving our understanding at the system level will allow us to build general brain-like structures or interfaces, with the intent to derive more-efficient and energy-frugal hardware systems and to develop effective cures for diseases or conditions, including acalculia and aphasia, that inhibit or limit brain function.

This review is useful for the sake of designing highly-scalable brain-like computing systems, but it is also critical as a stepping stone for making brain-computing interfaces feasible for the regions of the brain that control mathematical ability [40]. Even though limits of our understanding of the human brain abound, we can still learn from existing data to inspire the design of neuromorphic circuits with better likeness to their biological counterparts, greater energy-efficiency, and so on.

Much study and analysis remain to be done to create a full picture of arithmetic in the human brain. For example, one aspect that is missing in this paper is the signal-processing component, that is, the transformation of encoded neuronal firing sequences in the human brain from equivalently-valued input pulses to output arithmetic results. An analysis of biological neural connectivity for computational network-building, similar to Cannas' artificial network presentation [30], is also missing, and should be addressed in future work.

A number of other details need to be added to this discussion, including challenges in scaling existing CMOS neuromorphic solutions, given that we are currently limited in connectivity with model neurons and a biological brain "replica" is not feasible. Furthermore, there is evidence that mathematical training changes the neural pathways and regional activations associated with arithmetic, which adds a new evolutionary dimension to future research in this area [41].

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