Optimal number of disc clock tracks for block-oriented rotating associative processors

B. Parhami

Indexing terms: Digital computers and computation, Optimisation

Abstract: Block-oriented rotating associative processors provide a highly cost-effective solution to the need for parallel processing of large volumes of data in information storage and retrieval applications. Such processors are implemented by incorporating some processing logic into each read/write head of a fixed-head (head-per-track) disc memory. With non-self-clocked data recording, which offers higher storage densities, the usual data storage format is to assign the fixed capacity dictated by the maximum bit density and the length of the innermost (shortest) track to all tracks, thus making it possible to use a single clock track for the synchronisation of read and write operations. In the paper, we note that with a single clock track about half of the total recording capacity of the disk surface is wasted. This leads to the proposal for dividing the tracks into groups, with one clock track for each group. It is shown that with optimal design, 90–95% of the useful disk capacity can be utilised. This represents a 70–90% improvement over the case of a single clock track and is achieved at virtually no extra cost.

1 Introduction

Associative (content-addressable) memories and processors [2, 4, 9] have been the subject of much study for over three decades. However, even with recent advances in VLSI technology the construction of a fully parallel associative device with the capacity to hold a large database is still impractical. An alternative to the fully parallel approach is provided by the block-oriented organisation [7] which can be implemented by incorporating some processing logic into each read/write head of a fixed-head (head-per-track) disc [6]. Many such systems have been implemented or proposed over the past 15 years.

There are two classes of methods for storing data on magnetic disks [1, 5]. In self-clocked schemes, the data and timing signals are intermixed so that read/write mechanisms can function properly without a need for external timing control. These schemes have developed into sophisticated run-length-limited (RLL) codes for data recording [3, 8]. With non-self-clocked formats, an external timer which is synchronised by timing signals stored on a clock track provides the needed timing control for the read/write circuitry. These latter schemes require fewer changes in the direction of disc surface magnetisation and thus can support higher recording densities. The usual arrangement for non-self-clocked recording on fixed-head discs is to use a single clock track, thus effectively assigning a fixed data capacity to all tracks. This fixed capacity is dictated by the maximum recording density and the length of the innermost (shortest) track and is thus wasteful of space in the outer tracks.

In this paper, we note that with a single clock track only 50–53% of the total recording capacity of the disc surface of a block-oriented rotating associative processor is utilised. This leads to the proposal for dividing the tracks into equal-capacity groups, with one clock track for each group. It is shown that with optimal design, 90–95% of the useful disc capacity can be utilised. This represents a 70–90% improvement over a single clock track and is achieved at virtually no extra cost. Since in block-oriented rotating associative processors, disc tracks are processed independently and in parallel, they can have arbitrary capacities. This is in contrast with applications of disc memories in general-purpose systems where completely variable track capacities may cause difficult memory-management problems. However, in any application where disc blocks are accessed through direct external pointers (e.g. paging discs) the optimal design discussed in this paper may be applicable.

2 Notation

The following notation is used throughout this paper. When parameters can be computed from others in a straightforward manner, equivalent expressions are also given. Other relationships will be explored in subsequent sections of the paper. Fig. 1 depicts some of the parameters of interest.

- u = unit of length used in this paper = distance between centrelines of adjacent tracks = 1/track density
- d = bit density/track density = number of bits that can be recorded in unit length u (typically $d \gg 1$)
- T_i = ith track on a disc surface; T_1 is the outermost, T_t is innermost
- t = number of tracks on disc surface = $r_1 - r_t + 1$
- r_i = integer denoting radius of ith track on disc surface = $r_1 - i + 1$
- c_i = bit capacity of ith track on disc surface = $2\pi d r_i$; we will assume that $2\pi d$ is an integer
- Γ_j = jth track group: a number of contiguous, equal-capacity tracks sharing a clock track; Γ_1 is outermost
- γ = number of track groups on disc surface = number of clock tracks

Paper 6865E (C2), first received 17th May 1988 and in revised form 27th June 1989

The author is at the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106, USA

τ_j = number of tracks in jth track group, including clock track ($\tau_j > 1$, $\sum_{j=1}^{\gamma} \tau_j = t$)

ρ_j = radius of innermost track in jth track group = $r_1 + 1 - \sum_{k=1}^{j} \tau_k$

κ_j = bit capacity of the jth track group = $2\pi d \rho_j(\tau_j - 1)$ for one disc surface

C = total bit capacity of disc = $\sum_{j=1}^{\gamma} \kappa_j$ for one disc surface $< \sum_{i=1}^{t} c_i$ due to clock tracks

U = utilisation ratio: fraction of upper bound for total disc recording capacity actually utilised

$\{\ldots\}$ = parameters or assumptions used in computing preceding value; e.g. $C\{r_1, r_t\}$, $U\{\gamma : 1\}$

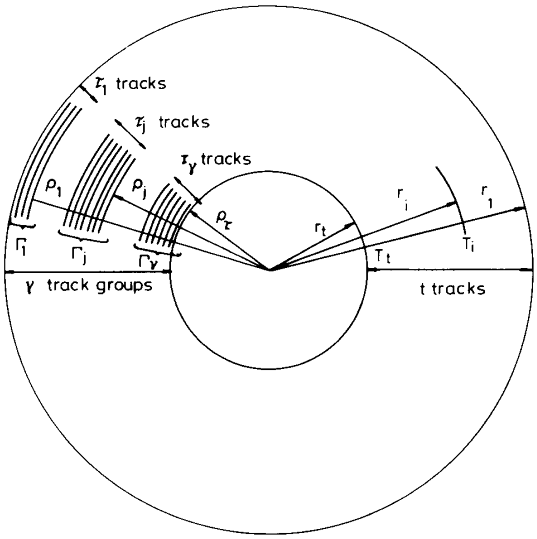

Fig. 1 *Some notation and disc parameters*
All parameters are integers

In addition, the superscripts *max*, *opt* and *ub* identify maximal, optimal, and upper-bound values, respectively.

3 Discs with single clock track

In this Section we try to answer the following question for discs having a single recording surface and a single clock track ($s:1, \gamma:1$): given a fixed outer track radius, what is the optimal number of tracks? Intuitively, having too few tracks will waste disc surface whereas having too many tracks will make the innermost track very short, thus having an adverse effect on capacity. The following lemma gives us the optimal value for $t = r_1 - r_t + 1$.

Lemma 1: Given a fixed outer track radius r_1 and a single clock track (hence, equal capacities for all the $t - 1$ data tracks), the total disc capacity is maximized when the number of tracks and the inner track radius are chosen to be $t^{opt} = \lfloor r_1/2 \rfloor + 1$ and $r_t^{opt} = \lceil r_1/2 \rceil$, respectively.

Proof: The total disc capacity is $2\pi d r_t(t-1) = 2\pi d r_t (r_1 - r_t)$. Thus, $r_t = \lfloor r_1/2 \rfloor$ and $r_t = \lceil r_1/2 \rceil$ both yield the maximal capacity $C^{max}\{r_1, \gamma:1\} = 2\pi d \lfloor r_1/2 \rfloor \lceil r_1/2 \rceil$; i.e. $\pi d r_1^2/2$ or $\pi d(r_1^2 - 1)/2$, depending on whether r_1 is even or odd, respectively. The value $r_t = \lceil r_1/2 \rceil$ is preferable because it implies one less track (and thus one less read/write head) when r_1 is odd.

An upper bound on the disc capacity is obtained if we assume that data is recorded at the maximum possible density on all tracks without wasting any disc surface for clock tracks. Thus

$$C^{ub}\{r_1, r_t\} = \sum_{i=1}^{t} 2\pi d r_i = 2\pi d \sum_{k=r_1}^{r_t} k$$
$$= \pi d(r_1 + r_t)(r_1 - r_t + 1)$$

It is easily seen that the smaller the value of r_t, the larger the value of $C^{ub}\{r_1, r_t\}$; with the maximum value of $\pi d r_1(r_1 + 1) \cong \pi d r_1^2 = \pi d t^2$ assumed for $r_t = 1$. However, it is not cost-effective to reduce the value of r_t beyond a certain point, since the cost of the additional read/write mechanisms cannot be justified in view of the relatively small gain in capacity from the extremely short inner tracks.

The value of $C^{ub}\{r_1, r_t\}$ provides an 'ideal' reference point against which the efficiency of various schemes can be measured. For example, the efficiency of the optimal single-clock-track arrangement (given by Lemma 1) compared to the ideal upper bound is

$$U^{max}\{r_1, \gamma:1\} = C^{max}\{r_1, \gamma:1\}/C^{ub}\{r_1, r_t\}$$
$$\cong r_1^2/[2(r_1 + r_t)(r_1 - r_t + 1)]$$

Thus, $U^{max}\{r_1, \gamma:1\}$ ranges from about 50% for the practically unachievable extreme of $r_t = 1$ to about 67% when $r_t = r_1/2$. In practice, $r_t \leqslant r_1/4$ is quite feasible. Thus, at best only about 50–53% of the potential disc recording capacity is utilised with a single clock track.

4 Discs with two clock tracks

One way to improve the utilisation ratio U is to divide the tracks into $\gamma > 1$ groups, with each group having its own clock track. Even though more space will be wasted by the multiple clock tracks than in the single clock track case, one hopes that the added capacity for tracks in groups $1, 2, \ldots, \gamma - 1$ will more than compensate for the loss. We start with the simplest possible case; namely, with $\gamma = 2$.

Lemma 2: Given fixed outer and inner track radii r_1 and r_t, division of the $t = r_1 - r_t + 1$ tracks into groups of τ_1 and τ_2 tracks ($\tau_1 + \tau_2 = t$) can result in improved capacity only if $r_t < r_1 + 2 - 2\sqrt{(r_1 + 1)}$. Easily obtainable equivalent forms of this condition are $t > 2\sqrt{(r_1 + 1)} - 1$ and $t > 2\sqrt{(r_t)} + 1$.

Proof: The perimeter of the innermost track of the jth group is $2\pi\rho_j$. Thus using $\tau_2 = t - \tau_1$, we have

$$C\{\gamma:2\} = \kappa_1 + \kappa_2 = 2\pi d[\rho_1(\tau_1 - 1) + \rho_2(\tau_2 - 1)]$$
$$= 2\pi d[-\tau_1^2 + (r_1 - r_t + 2)\tau_1 + r_t(t - 1) - r_1 - 1]$$

Comparing the above with $C\{\gamma:1\} = 2\pi d r_t(t - 1)$, we conclude that $\gamma = 2$ improves the capacity over $\gamma = 1$ if

$$\tau_1^2 - (r_1 - r_t + 2)\tau_1 + r_1 + 1 < 0$$

A necessary condition for this inequality to hold is that the corresponding quadratic equation have two real roots. Thus, we must have $(r_1 - r_t + 2)^2 > 4(r_1 + 1)$ which is the desired condition.

Corollary 1: If in a disc with a single clock track r_t has the optimal value given by Lemma 1, then dividing the tracks into two groups is always advantageous.

Proof: From Lemma 1, we have $t^{opt} = \lfloor r_1/2 \rfloor + 1$. Examining the requirement for t in terms of r_1 from Lemma 2, we note that it is satisfied by t^{opt} for $r_1 \geqslant 10$; a condition which always holds in practice.

Lemma 3: Given fixed outer and inner track radii r_1 and r_t, division of the $t = r_1 - r_t + 1$ tracks into groups of τ_1 and τ_2 tracks ($\tau_1 + \tau_2 = t$) results in maximum capacity if $\tau_1 = \lceil t/2 \rceil$.

Proof: We rewrite the expression for $C\{\gamma : 2\}$ obtained in the proof of Lemma 2 as

$$C\{\gamma : 2\}/(2\pi d) = (t + 1)^2/4 + r_t(t - 1) - r_1 - 1 - [(t + 1)/2 - \tau_1]^2$$

To maximise $C\{\gamma : 2\}$, the last term in the above expression must be made as small as possible. Thus, if t is odd, we have $\tau_1^{opt} = (t + 1)/2 = \lceil t/2 \rceil$. If t is even, then $t/2 = \lceil t/2 \rceil$ and $t/2 + 1$ are equally optimal choices for τ_1.

Corollary 2: In dividing t tracks into two groups of τ_1 and τ_2 tracks, optimality is achieved if $0 \leqslant \tau_1 - \tau_2 \leqslant 1$.

Proof: If t is odd, then by Lemma 3 we have $\tau_1^{opt} = (t + 1)/2$ and $\tau_2^{opt} = (t - 1)/2$. If τ is even, then we can choose $\tau_1^{opt} = \tau_2^{opt} = t/2$.

Obviously, the results of Lemmas 2 and 3 must be applied together. Lemma 3 tells us how to divide optimally t tracks into two groups when we choose to divide and Lemma 2 tells us when it is advantageous to do the splitting. Even though these results are interesting in their own right, they were derived chiefly because they facilitate the proof of the subsequent main results.

5 Optimal number of clock tracks

We now consider the case of $\gamma > 1$ track groups. Our aim is to find the optimal number of track groups (clock tracks) for given outer and inner track radii r_1 and r_t.

Lemma 4: Given an arbitrary integer $\gamma(\gamma < \sqrt{2t})$, there exists an optimal division of t tracks into exactly γ track groups of $\tau_1, \tau_2, \ldots, \tau_\gamma$ tracks for which $\tau_1 = \lfloor t/\gamma + (\gamma - 1)/2 \rfloor$.

Proof: Consider an optimal division of the t tracks into γ track groups such that for all i ($1 \leqslant i < \gamma$), we have $0 \leqslant \tau_i - \tau_{i+1} \leqslant 1$. From corollary 2, such an optimal division must exist. Thus with τ_1 tracks in the outermost track group, t can be at most $\gamma\tau_1$ and it must be at least $\tau_1 + (\tau_1 - 1) + \cdots + (\tau_1 - \gamma + 1) = \gamma\tau_1 - \gamma(\gamma - 1)/2$. Equivalently given t tracks on the disc, τ_1 must satisfy

$$[t/\gamma] \leqslant \tau_1 \leqslant [t/\gamma + (\gamma - 1)/2]$$

Furthermore, the expression for C in the proof of Lemma 3 shows that it is advantageous to have $\tau_i - \tau_{i+1} = 1$ whenever possible. To allow this, τ_1 must be selected to have the maximum value in the above range. This last part of the argument can be formalised as follows. We will show that there exists an optimal arrangement for which $\tau_i - \tau_{i+1} = 1$, except possibly for a single value of the index i. Suppose that in some arrangement satisfying $0 \leqslant \tau_i - \tau_{i+1} \leqslant 1$ for all i ($1 \leqslant i < \gamma$), we have $\tau_i = \tau_{i+1}$ for two or more values of i. Let j and k be the smallest and largest of these values. Then we have

$$\tau_1 > \tau_2 > \cdots > \tau_j = \tau_{j+1} \geqslant \tau_{j+2} \geqslant \cdots \geqslant \tau_k = \tau_{k+1} > \tau_{k+2} > \cdots > \tau_t$$

We can reassign the tracks to groups so that Γ_j has $\tau_j + 1$ tracks and Γ_{k+1} has $\tau_{k+1} - 1$ tracks. If we show that this reassignment increases the disc capacity, the proof is complete. This is due to the fact that repeating this procedure will move j toward 1 and k toward t, until eventually one of them disappears at the edge. The change in capacity resulting from the above rearrangement can be attributed to four factors:

(*a*) $+2\pi d(\rho_j - 1)$: capacity gain as a result of the track added to Γ_j

(*b*) $-2\pi d(\tau_j - 1)$: capacity loss of old Γ_j data tracks because the innermost track is now $2\pi d$ shorter

(*c*) $-\sum_{i=j+1}^{k} 2\pi d(\tau_i - 1)$: capacity loss of Groups $\Gamma_{j+1}, \Gamma_{j+2}, \ldots, \Gamma_k$ resulting from their inward movement

(*d*) $-2\pi d\rho_{k+1}$: capacity loss as a result of the removed track of Γ_{k+1}

With some manipulation, the sum of the above four values can be written as $2\pi d[k - j - (\tau_i - \tau_{k+1})]$; but from the above assumptions, $\tau_j - \tau_{k+1} = \tau_{j+1} - \tau_k \leqslant k - (j + 1)$. Thus the capacity has improved by at least $2\pi d$.

Note that Lemma 4 can be applied recursively to construct a 'standard' optimal configuration with γ track groups from the possibly many optimal arrangements. In other words, once τ_1 is determined, the remaining $t - \tau_1$ tracks can be divided into $\gamma - 1$ track groups by using the same procedure. The condition $\gamma < \sqrt{2t}$ in the statement of Lemma 4 is needed for this recursive application of the result to make sense. Having proven all the necessary lemmas, we are now ready to present our main result in the following theorem.

Theorem 1: Given fixed outer and inner track radii r_1 and r_t, division of the $t = r_1 - r_t + 1$ tracks into γ track groups of $\tau_1, \tau_2, \ldots, \tau_\gamma$ tracks (by using Lemma 4) results in maximum total disc capacity if γ has the optimal value

$$\gamma^{opt}\{r_1, t\} = \lceil 2\sqrt{\{r_1 - t/2 + 24/24 - \sqrt{[(r_1 + 25/24)(r_1 - t + 25/24)]}\}} \rceil$$

Proof: Let $t/\gamma + (\gamma - 1)/2$ be an integer. This assumption, if not valid, will yield a value for $\gamma^{opt}\{r_1, r_t\}$ which is less than or equal to the actual value of $\gamma^{opt}\{r_1, r_t\}$. With this assumption, recursive application of Lemma 4 results in $\tau_j^{opt}\{r_1, r_t, \gamma\} = \tau_1^{opt} - j + 1$. The innermost track of track group j will have the radius $\rho_j^{opt}\{r_1, r_t, \gamma\} = r_1 + 1 - (t/\gamma + \gamma/2)j + j^2/2$. Thus, the total disc capacity will be

$$C^{opt}\{r_1, r_t, \gamma\} = 2\pi d \sum_{k=1}^{\gamma} [(\tau_k^{opt} - 1)\rho_k^{opt}]$$

$$= 2\pi d \sum_{k=1}^{\gamma} [t/\gamma + (\gamma - 1)/2 - k] \times [r_1 + 1 - (t/\gamma + \gamma/2)k + k^2/2]$$

Multiplying out and using the identities $\sum_{k=1}^{\gamma} k = \gamma(\gamma + 1)/2$, $\sum_{k=1}^{\gamma} k^2 = \gamma(\gamma + 1)(2\gamma + 1)/6$, and $\sum_{k=1}^{\gamma} k^3 = \gamma^2(\gamma + 1)^2/4$, we arrive at the following result:

$$C^{opt}\{r_1, r_t, \gamma\} = \pi d[\gamma^3/12 - (2r_1 - t + 25/12)\gamma + t(2r_1 - t + 3) - t^2/\gamma]$$

To determine the optimal value for γ, we find the first and second-order derivatives of C^{opt} with respect to γ.

$$\partial C^{opt}\{r_1, r_t, \gamma\}/\partial\gamma = \pi d[\gamma^2/4 - (2r_1 - t + 25/12) + t^2/\gamma^2]$$

$$\partial^2 C^{opt}\{r_1, r_t, \gamma\}/\partial\gamma^2 = \pi d(\gamma/2 - 2t^2/\gamma^3)$$

For $\gamma < \sqrt{2t}$, the second-order derivative of $C^{opt}\{r_1, r_t, \gamma\}$ is always negative. As a result, the smaller solution of $\partial C^{opt}\{r_1, r_2, \gamma\}/\partial\gamma = 0$ (which satisfies $\gamma < \sqrt{2t}$) maximises the capacity. This solution yields the result we are seeking for $\gamma^{opt}\{r_1, t\}$. In general, the above value for $\gamma^{opt}\{r_1, t\}$ is not an integer. Thus, as stated previously, the next largest integer must be selected to achieve optimality.

Fig. 2 shows the values of $\gamma^{opt}\{r_1, t\}$ for different values of r_1 and t. Curves corresponding to some values of $\gamma^{opt}\{r_1, t\}$ (e.g. 5, 7, 8, 9, 11, ...) have been omitted for clarity. The utilisation ratio with the optimal number of clock tracks is

$$U^{max}\{r_1, r_t, \gamma^{opt}\{r_1, t\}\} = [\gamma^3 - (24r_i - 12t + 25)\gamma + 12t(2r_1 - t + 3) - 12t^2/\gamma]/[12(r_1 + r_t)(r_1 - r_t + 1)]$$

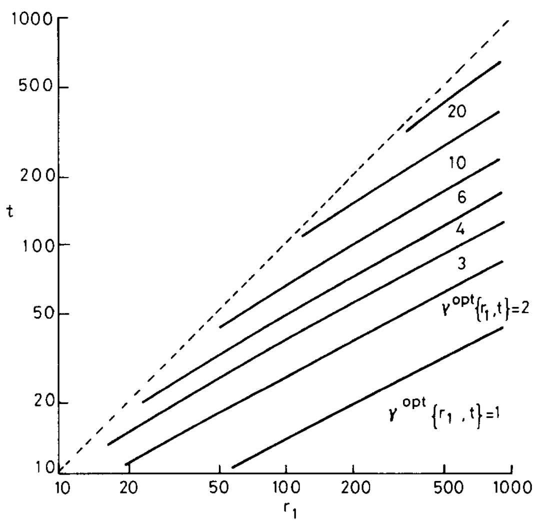

Fig. 2 *Optimal number of clock tracks*

The value of $U^{max}\{r_1, r_t : 1, \gamma^{opt}\{r_1, t\}\}$, which represents a pessimistic lower bound for $U^{max}\{r_1, r_t, \gamma^{opt} r_1, t\}\}$, has bee plotted in Fig. 3 for different values of r_1 and t. It is

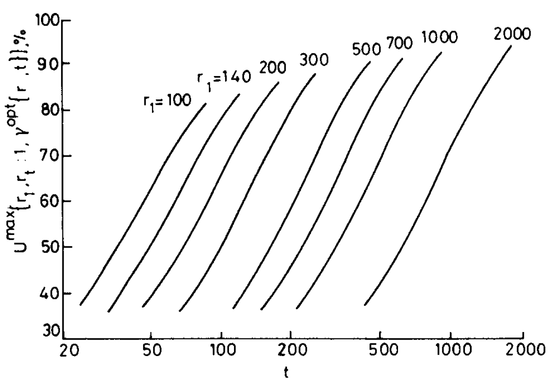

Fig. 3 *Lower bound for optimal utilisation ratio*

seen that even this pessimistic lower bound can exceed 90% for certain optimal configurations.

Examples: For a disc with $r_1 = 100$ and $t = 60$, we have from theorem 1, $\gamma^{opt}\{r_1 : 100, t : 60\} = 6$ and from Lemma 4, $\tau_1(t : 60, \gamma : 6) = 12$. The remaining 48 tracks must be divided into 5 groups. Repeated use of Lemma 4 yields the complete standard optimal division: 12, 11, 10, 10, 9, 8. In the case of a disc with $r_1 = 200$ and $t = 125$, theorem 1 yields $\gamma^{opt}\{r_1 : 200, t : 125\} = 8$, with optimal dividion being: 19, 18, 17, 16, 15, 14, 13, 13.

6 Conclusion

We have considered fixed-head discs used in the design of block-oriented rotating associative processors and have shown that the disc surface utilization can be improved by 70–90% through the incorporation of multiple clock tracks in an optimal way. The provision of additional clock tracks does not affect the system cost, as the read-write heads for these tracks are already present in a head-per-track disc and the processing cells such as those designed for the RAPID block-oriented associative processor [6] can be utilised with no modification. In addition, the provision of multiple clock tracks makes the system more robust in the sense that a bad clock track or failure of the associated read head will only result in partial loss of capacity. This method may also be useful for other applications of fixed-head discs where variable track capacities do not result in difficult memory management problems (e.g. paging discs).

This study can be extended in several directions. If data on the disc are stored and retrieved in terms of B-bit data blocks, the division of the tracks into two groups is worthwhile only if the extra capacity of tracks in group 1 is greater than or equal to the size of a data block. Another type of constraint arises when it is required that a byte of data be read out in parallel (see, e.g., [6]). In this case, we may assign groups of b contiguous tracks to hold the data in a byte-parallel format. Thus, we must have $\tau_i = 1 \bmod b$, so that each track group contains a clock track plus an integral number of 'super-tracks' (each composed of b contiguous tracks). The effects of these additional constraints and of having multiple recording surfaces on the optimal number of clock tracks are currently being investigated.

7 References

1 FRENCH, M.: 'Rotating disks and drums set peripheral memories spinning', *in* RILEY, W. B. (Ed.): *Electronic Computer Memory Technology* (McGraw-Hill, 1971), pp. 143–149

2 HANLON, A. G.: 'Content-addressable and associative memory systems: a survey', *IEEE Trans Electron Comput.*, 1966, **EC-15**, (4), pp. 509–521

3 KANEKO, S.: 'Magnetic disk drive', *in* SAKURAI, Y. (Ed.): 'Recent Magnetics for Electronics (Japan Annual Reviews in Electronics, Computers, & Telecommunications' (North-Holland, 1983), vol. 10, pp. 87–100

4 KOHONEN, T.: 'Content-addressable memories' (Springer-Verlag, 1987), 2nd Edition

5 LANGHOLZ, G., FRANCIONI, J., and KANDEL, A.: 'Elements of computer organization' (Prentice-Hall, Englewood Cliffs, 1989), p. 203

6 PARHAMI, B.: 'A highly parallel computing system for information retrieval'. AFIPS Conf. Proc., Vol. 41 (1972 Fall Joint Computer Conf.) AFIPS Press, Montvale, NJ, pp. 681–690

7 PARHAMI, B.: 'Associative memories and processors: an overview and selected bibliography', *Proc. IEEE*, 1973, **61**, (6), pp. 722–730

8 SIEGEL, P.H.: 'Recording codes for digital magnetic storage', *IEEE Trans. Magnetics*, 1985, **MAG-21**, (5), pp. 1344–1349

9 YAU, S.S., and FUNG, H.S.: 'Associative processor architectures: a survey', *Comput. Survey.*, 1977, **9**, (1), pp. 3–27