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Pruned three-dimensional toroidal networks

Ding-Ming Kwai, Behrooz Parhami *

Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106-9560, USA

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Abstract

Using a unified framework, we show that a pruned 3D torus can be explicitly specified by its pruning direction along which the links are uniformly removed from a complete 3D torus. The resulting networks, that offer the advantages of lower node degree and simpler layout, are inherently node-symmetric and maintain the same diameter as the original torus. It turns out that the simplest pruning scheme is also best in preserving regularity, symmetry, and performance properties of the torus.

Keywords: Cayley graph; Interconnection networks; Manhattan street network; Processor array; Pruning strategy; Torus

1. Introduction

The links of a richly connected network can be removed in a periodic fashion for reduced complexity and hence increased performance. Such incomplete networks derived from pruning 3D meshes have been shown to be quite effective [5,9]. Let us denote the forward/backward connections along each of the three dimensions X, Y, Z with $+/-$ signs. A simple pruning scheme, used in the Tera MTA interconnection network [1], is to alternately remove the $\pm X$ and $\pm Y$ links along dimension Z so that the node degree is reduced from six to four. (We refer to this type of network as T_1 in Section 2.)

Pruning a 3D torus, as done in [1], has advantages over pruning a 3D mesh in view of the node symmetry resulting from the inclusion of wraparound links. Except for the $2 \times 2 \times 2$ torus, whose pruned version degenerates into a ring of eight nodes, the longest distance between nodes, or the diameter, remains the same as that of the complete 3D torus, while the average internode distance increases only slightly. The negative effect of the bisection width being reduced to half can be more than compensated for by the increased channel capacity and more compact layout.

The same pruning scheme is equally applicable to the unidirectional version of a torus, known as Manhattan street network [4,8]. In fact, the Manhattan street network can itself be viewed as a pruned torus if each bidirectional link in the torus consists of two unidirectional links. More complicated pruning schemes may be devised for both bidirectional and unidirectional tori by arranging the link removals along other directions. In essence, these variations are manipulations of connections within the Euclidean 3-space and the resulting networks share common properties.

Pruning higher-dimensional tori, and variants that replace the point-to-point connections along each dimension with buses, have been shown in [6] and [10], respectively. In this paper, we focus on the bidirec-

* Corresponding author. Email: parhami@ece.ucsb.edu.

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tional 3D torus and unveil several interesting topological properties related to such pruned 3D tori. Our presentation is organized as follows. Section 2 contains the definitions of the various types of pruned tori. Section 3 proves that the resulting networks belong to the class of Cayley graphs and thus with wraparound links included, they become not only regular but also node-symmetric. Section 4 includes results on the diameter and average internode distance of the pruned tori. Section 5 contains our conclusion.

2. Preliminaries

Let us denote each node in the 3D k-torus as (x, y, z), where $0 \leqslant x, y, z \leqslant k-1$. In a complete 3D k-torus, each node (x, y, z) is connected to six neighbors, namely, $(x \pm 1, y, z)$, $(x, y \pm 1, z)$, and $(x, y, z \pm 1)$. Here and throughout, it will be understood that all node-index expressions are calculated modulo k. Furthermore, we require k to be an even number to ensure that the pruned torus is regular and of degree four.

Consider two types of pruned tori, T_1 and T_2, defined as follows. For these pruned 3D k-tori considered in this paper, each node (x, y, z) is connected to its two neighbors $(x, y, z \pm 1)$ along dimension Z. In addition, each node (x, y, z) in T_1 is also connected to

$$\begin{cases} (x \pm 1, y, z) & \text{if } z = \text{even}, \\ (x, y \pm 1, z) & \text{if } z = \text{odd} \end{cases}$$

and in T_2 to

$$\begin{cases} (x+1, y, z) \text{ and } (x, y+1, z) & \text{if } x+y+z = \text{even}, \\ (x-1, y, z) \text{ and } (x, y-1, z) & \text{if } x+y+z = \text{odd}. \end{cases}$$

The conditions for the X and Y connections in T_1 and T_2 define the pruning directions as z and $x + y + z$, respectively. It is easy to verify that both pruned networks are regular and of degree four. Furthermore, both are Hamiltonian, meaning that they contain a ring, encompassing all the nodes, as a subgraph. Constructive proof of Hamiltonicity is quite simple and is omitted here.

Fig. 1 shows examples of pruned $4 \times 4 \times 4$ tori T_1 and T_2, where the nodes are shaded if their positions along the pruning direction are odd-numbered. The wraparound links are not shown to avoid clutter.

3. Symmetry properties

Let $*$ be an associative binary operator and Ω be a set of generators for a finite group Γ ($\Omega \subseteq \Gamma$) such that

(i) the identity $\iota \notin \Omega$;
(ii) if $\omega \in \Omega$, then its inverse $\omega^{-1} \in \Omega$, where $\omega * \omega^{-1} = \iota$.

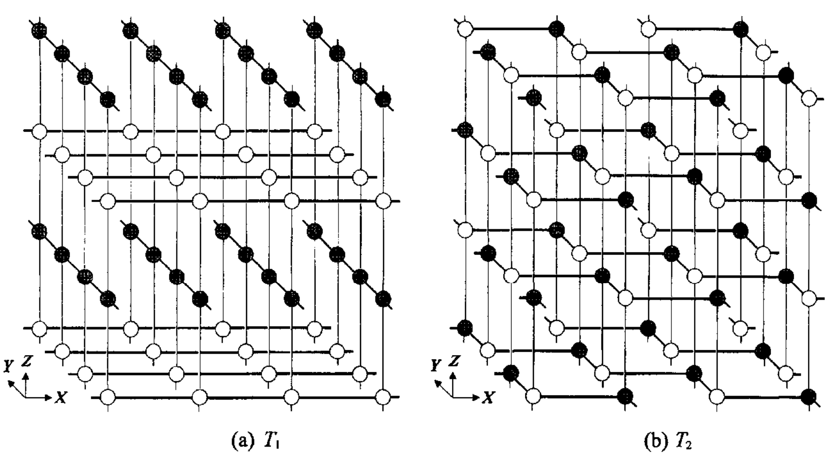

Fig. 1. Pruned 3D 4-tori obtained by pruning along (a) the z direction and (b) the $x + y + z$ direction.

In our case, the finite group is the node set of the 3D k-torus,

$$\Gamma = \{(x, y, z)\colon 0 \leqslant x, y, z \leqslant k-1\}.$$

A Cayley graph is defined as a graph whose nodes α and $\alpha * \omega (\alpha, \alpha * \omega \in \Gamma)$ are connected if and only if $\omega \in \Omega$. The cardinality of the generator set Ω determines the node degree. It is well known that Cayley graphs are node-symmetric [2,3]. For a detailed discussion on the symmetry properties of Cayley graphs, we refer the reader to [3,7].

Theorem 1. *The pruned* 3D *k-tori T_1 and T_2 are Cayley graphs.*

Proof. To facilitate our discussion, node indices will be expressed as 3-vectors. Let $[x, y, z]^{\mathrm{T}} \in \Gamma$ and $[a, b, c]^{\mathrm{T}} \in \Omega$. Define the group operator $*$ as a semi-directed product.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} * \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \Phi^{f(x,y,z)} \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

where

$$\Phi = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

a permutation matrix with the periodic property $\Phi^2 = I$, is raised to the integer power $f(x, y, z)$. In other words, $\Phi^{f(x,y,z)}$ is either the identity matrix or Φ, based on the chosen function f and the values of the coordinates x, y and z. As before, the calculations are performed component-wise modulo k.

In each case, we specify the pruning direction $f(x, y, z)$ and derive the associated generator set Ω whose cardinality is equal to four.

(i) T_1: $f(x, y, z) = z$ and

$$\Omega = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

(ii) T_2: $f(x, y, z) = x + y + z$ and

$$\Omega = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

The identity ι can be easily shown to be $(0, 0, 0)$. The proof is complete by noting that the derived generator sets are closed under inverse, thus making the links bidirectional. □

We note that there does not exist a pruning direction $f(x, y, z)$ that combines two out of the three directions, since such selections leave the resulting networks unconnected. Fig. 2 shows an example that pruning along $x + y$ makes the shaded and non-shaded sub-networks disjoint. As a consequence of Theorem 1, an exhaustive check through all possibilities confirms that permuting X, Y, and Z dimensions leads to networks that are isomorphic to either T_1 or T_2. Permuting the X and Z dimensions, while still maintaining the same generator set as for T_1, leads to Theorem 2.

Theorem 2. *The pruned* 3D *k-torus T_1 is edge-symmetric.*

Informally, the edge symmetry makes all links look alike. This property distinguishes T_1 from T_2. Observe that the $\pm Z$ link in T_2 is always within a cycle of length k, but there is no such cycle for $\pm X$ or $\pm Y$ links. Thus, the pruned 3D k-torus T_2 is not edge-symmetric.

4. Diameter and average distance

Theorem 3. *For $k \geqslant 4$, the diameter of pruned* 3D *k-tori T_1 and T_2 is exactly* $3k/2$.

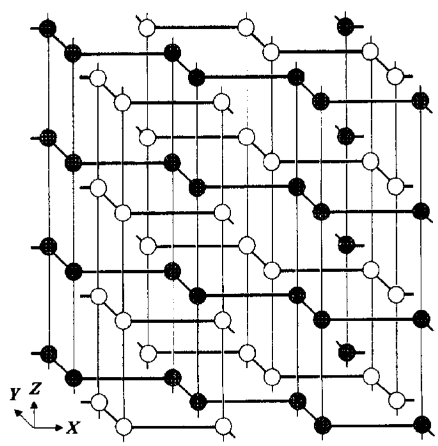

Fig. 2. Pruned 3D 4-torus obtained by pruning along the $x + y$ direction.

Proof. Before deriving the diameter, it is helpful to discuss a routing algorithm for such networks. The routing algorithm is based on clustering pairs of nodes into groups along the Z dimension. Each group possesses a complete set of dimensional links and allows routing in all directions. Then the pruned 3D k-torus can be viewed as a complete 3D torus with $k \times k \times k/2$ nodes. Recall that k is an even number.

The number of routing steps taken along dimensions X and Y is at most $k/2 + k/2 = k$. The worst case needs $\lfloor k/4 \rfloor$ steps to move between groups along dimension Z and $\lceil k/4 \rceil$ steps to move within groups. The diameter is thus upper bounded by $k + \lfloor k/4 \rfloor + \lceil k/4 \rceil = 3k/2$. Since T_1 and T_2 are subgraphs of the complete 3D k-torus with diameter $3k/2$, this is also a lower bound for the diameter. □

Theorem 3 states that the two pruned 3D k-tori T_1 and T_2 have the same diameter. Again, we can distinguish them by the difference between their average distances. The average distance of T_1 can be determined analytically.

Theorem 4. *The average distance of the pruned* 3D *k-torus T_1 is* $3k/4 + 2(k^{-1} - k^{-2})$.

Proof. Without loss of generality, we select node $(0, 0, 0)$ as the source and route from this node to every other node (x, y, z) along a shortest path. Consider the increase in the sum of distances compared to $3k^4/4$ for the complete 3D k-torus. Two extra routing steps need to be taken if the destination node (x, y, z) has $y \neq 0$ and $z = 0$. For all other cases, the shortest path in T_1 is of the same length as is in the complete torus. In the 3D k-torus, there are $k^2 - k$ nodes with $y \neq 0$ and $z = 0$. Thus the sum of distances is increased by $2(k^2 - k)$, leading to the average distance $3k/4 + 2(k^{-1} - k^{-2})$. □

Table 1 lists the diameter values and average distances of the pruned 3D k-tori T_1 and T_2 with k chosen as successive powers of two from 4 to 64. We have not been able to find a closed-form expression for the average distance of T_2, but curve fitting leads to a slope of $0.86k$ (compared to $0.75k$ for the complete 3D k-torus and $0.7513k$ for T_1). The advantage of T_1 over T_2 in terms of average internode distance grows as k increases.

Table 1
Diameter d and average distance $\bar{d}$ of pruned 3D k-tori

k	$d(T_1) = d(T_2)$ $= d(\text{Torus})$	$\bar{d}(\text{Torus})$	$\bar{d}(T_1)$	$\bar{d}(T_2)$
4	6	3	3.38	3.31
8	12	6	6.22	6.83
16	24	12	12.12	13.74
32	48	24	24.06	27.52
64	96	48	48.03	55.07

5. Conclusion

The pruned networks considered in this paper are subgraphs of 3D tori in which some of the links have been removed from nodes. Previous work has demonstrated that such networks offer advantages in terms of reduced node complexity [9] and allowing low-cost implementation for domain decomposition [5]. We have shown that by removing links in a periodic fashion, desirable topological properties such as symmetry and Hamiltonicity can be preserved, with no penalty in diameter and with negligible increase in average internode distance, particularly for large k. It is interesting that the simpler, more intuitive architecture T_1 is better than T_2, in terms of both regularity and performance.

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