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ECE 140 Final Exam Solutions

Time: 3 Hours. **Instructions**: Do all four problems. You may use your text, notes and a calculator.

(20 points) 1. A two-dimensional density function is given by

$$f_{\tilde{x},\tilde{y}}(x,y) = \begin{cases} c_1 \cos(x+y)u(x)u(y), & 0 < x+y < \pi/2 \\ 0, & otherwise \end{cases}$$

(a) Find the normalization constant c_1 such that $f_{\tilde{x},\tilde{y}}(x,y)$ is a valid probability density function.

(b) Find the marginal densities $f_{\tilde{x}}(x),\ f_{\tilde{y}}(y)$. Are x and y independent r.v.s? Why or why not?

Answer

(a)

$$\int_0^{\pi/2} dx \int_0^{\pi/2-x} dy \cos(x+y)$$
$$= \int_0^{\pi/2} dx \left[\sin(x+\pi/2-x) - \sin(x)\right]$$
$$= \int_0^{\pi/2} dx \left[1-\sin(x)\right] = \frac{\pi}{2} - \left[-\cos(\pi/2)+\cos(0)\right]$$
$$= \pi/2 - 1$$
$$\Rightarrow c_1 = \frac{1}{\pi/2-1} = 1.7519$$

(b)

$$f_{\tilde{x}}(x) = \int f_{\tilde{x}\tilde{y}}(x,y)dy = c_1 \int_0^{\pi/2-x} dy \cos(x+y)$$
$$= c_1 \sin(x+y)\Big|_{y=0}^{\pi/2-x}$$
$$= c_1(1-\sin(x)), \quad 0 < x < \pi/2$$

$$f_{\tilde{y}}(y) = c_1 \int_0^{\pi/2-x} dx \cos(x+y)$$
$$= c_1(1-\sin(y)),\ 0 < y < \pi/2$$

x and y are clearly not independent, since $f(x, y) \neq f(x) f(y)$.

(20 points) 2. Let a random process be defined by

$$\tilde{x}(t) = \tilde{y}\exp(-t)u(t) + \tilde{w}\exp(-2t)u(t)$$

The random variables y and w are independent Gaussian, both with zero mean and unit variance.

(a) Find the mean function and autocorrelation function of x(t), $E\{\tilde{x}(t)\}, E\{\tilde{x}(t_1)\tilde{x}(t_2)\}$. Is x(t) wide-sense stationary? Explain.

(b) Find the average magnitude-squared short-time Fourier Transform of x(t), $E\{|\tilde{X}_T(\omega)|^2\}$. Recall this is

$$E\{|\tilde{X}_T(\omega)|^2\} = E\left\{\left|\int_{-T}^{T}\tilde{x}(t)e^{-j\omega t}dt\right|^2\right\}$$

Express your result in closed-form in terms of purely real-valued quantities. Using your result, what is the power spectral density $S_{xx}(\omega) = \lim_{T\to\infty}\frac{1}{2T}E\{|\tilde{X}_T(\omega)|^2\}$ of x(t)?

Answer

(a)

$$E\{\tilde{x}(t)\} = 0,\ \textit{linearity}\ E\{\tilde{y}\} = E\{\tilde{w}\} = 0$$
$$E\{\tilde{x}(t_1)\tilde{x}(t_2)\} = E\{(\tilde{y}\exp(-t_1)u(t_1) + \tilde{w}\exp(-2t_1)u(t_1))(\tilde{y}\exp(-t_2)u(t_2) + \tilde{w}\exp(-2t_2)u(t_2))\}$$
$$= E\{\tilde{y}^2\}\exp(-(t_1+t_2))u(t_1)u(t_2) + E\{\tilde{w}^2\}\exp(-2(t_1+t_2))u(t_1)u(t_2)$$
$$(\textit{Note}\ E\{\tilde{x}\tilde{y}\} = 0)$$
$$E\{\tilde{w}^2\} = E\{\tilde{y}^2\} = 1$$
$$\Rightarrow E\{\tilde{x}(t_1)\tilde{x}(t_2)\} = \exp(-(t_1+t_2))u(t_1)u(t_2) + \exp(-2(t_1+t_2))u(t_1)u(t_2)$$

x(t) is not w.s.s., since $R_{xx}(t_1, t_2) \neq R_{xx}(t_1 - t_2)$.

(b)

$$\tilde{X}_T(\omega) = \int_{-T}^{T} \left[\tilde{x}e^{-t}u(t) + \tilde{y}e^{-2t}u(t)\right] e^{-j\omega t} dt$$

$$= \tilde{x}\int_0^T e^{-t(1+j\omega)} dt + \tilde{y}\int_0^T e^{-t(2+j\omega)} dt$$

$$= \tilde{x}\frac{1-e^{-T(1+j\omega)}}{1+j\omega} + \tilde{y}\frac{1-e^{-T(2+j\omega)}}{(2+j\omega)}$$

$$E\left\{|\tilde{X}_T(\omega)|^2\right\} = E\left\{\left|\tilde{x}\frac{1-e^{-T(1+j\omega)}}{1+j\omega} + \tilde{y}\frac{1-e^{-T(2+j\omega)}}{(2+j\omega)}\right|^2\right\}$$

$$= \left|\frac{1-e^{-T(1+j\omega)}}{1+j\omega}\right|^2 + \left|\frac{1-e^{-T(2+j\omega)}}{2+j\omega}\right|^2$$

$$\left|\frac{1-e^{-T(1+j\omega)}}{1+j\omega}\right|^2 = \frac{1-2e^{-T}\cos(T\omega)+e^{-2T}}{1+\omega^2}$$

$$\Rightarrow \lim_{T\to\infty} E\left\{|\tilde{X}_T(\omega)|^2\right\} = \frac{1}{1+\omega^2} + \frac{1}{4+\omega^2}$$

$$\Rightarrow S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left\{|\tilde{X}_T(\omega)|^2\right\} = 0$$

(20 points) 3. A bandpass power spectral density corresponding to a w.s.s. process n(t) is defined by

$$S_{nn}(\omega) = \begin{cases} \exp(-|\omega-100|), & |\omega-100|<10 \\ \exp(-|\omega+100|), & |\omega+100|<10 \\ 0, & otherwise \end{cases}$$

(a) Find the equivalent low-pass power spectral density.

(b) Find the autocorrelation function of the low-pass equivalent process z(t).

Answer

(a) Note that $S_{xy}(\omega) = 0$ from the symmetry of the spectral density about $\omega_0 = 100$.

$$S_{nn}(\omega) = \frac{1}{2}S_{xx}(\omega-\omega_0) + \frac{1}{2}S_{xx}(\omega+\omega_0)$$

$$\Rightarrow S_{xx}(\omega) = 2\exp(-|\omega|), \ |\omega|<10$$

(b)

$$R_{zz}(\tau) = 2R_{xx}(\tau) + 2jR_{xy}(\tau), \quad R_{xy}(\tau) = 0$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-10}^{10} \exp(-|\omega|)e^{j\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{0}^{10} e^{\omega(j\tau-1)}d\omega + \frac{1}{2\pi}\int_{-10}^{0} e^{j\omega(j\tau+1)}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{j\tau-1}e^{10(j\tau-1)} - \frac{1}{j\tau-1}\right] + \frac{1}{2\pi}\left[\frac{1}{j\tau+1} - \frac{1}{j\tau+1}e^{-10(j\tau+1)}\right]$$

(20 points) 4. The difference equation characterizing a discrete-time linear system driven by white noise x(n) is

$$\tilde{y}(n) = .5\tilde{y}(n-1) - .04\tilde{y}(n-2) + \tilde{x}(n)$$

where $E\{\tilde{x}(n)\} = 0,\ E\{\tilde{x}(n)\tilde{x}(m)\} = \delta_{n,m}$. Assume the system is in steady-state.

(a) Find the power spectral density of the output y(n) in the Z-domain, $S_{yy}(z)$.

(b) Find the autocorrelation function $R_{yy}(k)$.

Answer

(a)

$$Y(z) = .5z^{-1}Y(z) - .04z^{-2}Y(z) + X(z)$$

$$H(z) = \frac{1}{1-.5z^{-1}+.04z^{-2}} = \frac{1}{(1-.1z^{-1})(1-.4z^{-1})}$$

$$S_{yy}(z) = \frac{1}{(1-.1z^{-1})(1-.4z^{-1})}\frac{1}{(1-.1z)(1-.4z)}$$

(b)

$$S_{yy}(z) = \frac{1}{(1-.1z^{-1})(1-.4z^{-1})}\frac{1}{(1-.1z)(1-.4z)}$$

$$= \frac{A}{(1-.1z^{-1})} + \frac{B}{(1-.4z^{-1})} + \frac{C}{(1-.1z)} + \frac{D}{(1-.4z)}$$

$$A = \frac{1}{(1-.4/.1)}\frac{1}{(1-.1(.1))(1-.4(.1))} = -.3507$$

$$B = \frac{1}{(1-.1/.4)}\frac{1}{(1-.1(.4))(1-.4^2)} = 1.6534$$

$$R_{yy}(k) = 1.6534(.4)^k - .3507(.1)^k, \;\; k \geq 0$$

$$R_{yy}(k) = 1.6534(.4)^{|k|} - .3507(.1)^{|k|}$$