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# ECE 158 Final Exam Solutions

**Time:** 3 hours. **Instructions:** Do all five problems, you may use your notes, text, and a scientific calculator.

**(20 points) Problem 1:** An LTI system has the following frequency response.

$$H(e^{j\omega}) = \begin{cases} \exp\left(-|\omega|\right), & |\omega| \le \pi/2 \\ 0, & \pi/2 < |\omega| \le \pi \end{cases}$$

(a) Find the impulse response $h(n) = F^{-1}\left(H(e^{j\omega})\right)$ where $F^{-1}(\ )$ is the inverse Discrete Time Fourier Transform. Express h(n) in a reasonably compact form, but you may leave the answer in the form $h(n) = \mathrm{Re}\{f(n)\}$ where $f(n)$ is some function of discrete time.

(b) Find the energy, E, in the impulse response in as compact a form as possible. Note that energy E is defined by

$$E = \sum_{n=-\infty}^{\infty} h(n)^2$$

**Answer**

(a)

$$h(n) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \exp\left(-|\omega|\right)\exp(j\omega n)\,d\omega$$

$$= \frac{1}{2\pi}\int_{0}^{\pi/2} \exp\left(\omega(jn-1)\right)d\omega + \frac{1}{2\pi}\int_{-\pi/2}^{0} \exp\left(\omega(jn+1)\right)d\omega$$

$$= \frac{\exp((\pi/2)(jn-1))-1}{2\pi(jn-1)} + \frac{1-\exp(-(\pi/2)(jn+1))}{2\pi(jn+1)}$$

$$= \frac{1-\exp((\pi/2)(jn-1))}{2\pi(1-jn)} + \frac{1-\exp((\pi/2)(-jn-1))}{2\pi(1+jn)}$$

$$= \mathrm{Re}\left\{\frac{1-e^{-\pi/2}\exp(j(\pi/2)n)}{\pi(1-jn)}\right\}$$

*Alternatively:*

$$= \frac{1+jn-jne^{-\frac{\pi}{2}}e^{j\frac{\pi}{2}n} - e^{-\frac{\pi}{2}}e^{j\frac{\pi}{2}n}}{2\pi(1+n^2)} + \frac{1-jn+jne^{-\frac{\pi}{2}}e^{-j\frac{\pi}{2}n} - e^{-\frac{\pi}{2}}e^{-j\frac{\pi}{2}n}}{2\pi(1+n^2)}$$

$$= \frac{1}{\pi(1+n^2)}\left(1-e^{-\frac{\pi}{2}}\cos\left(\frac{\pi}{2}n\right)+ne^{-\frac{\pi}{2}}\sin\left(\frac{\pi}{2}n\right)\right)$$

(b) Use Parseval's theorem.

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})\right|^2 d\omega = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} |\exp\left(-|\omega|\right)|^2 \, d\omega = \frac{1}{\pi}\int_{0}^{\pi/2} \exp\left(-2|\omega|\right) d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2} \exp\left(-2\omega\right) d\omega$$

$$\Rightarrow E = \frac{1-\exp(-\pi)}{2\pi}$$

**(20 points) Problem 2:** A causal LTI system has the following Z transfer function.

$$H(z) = \frac{1+.5z^{-1}}{(1-z^{-1}+.1875\ z^{-2})}$$

(a) Find the impulse response $h(n)$ of the above system. Show your work.

(b) Draw the signal flow graph of a Direct Form II implementation of the above system $H(z)$.

**Answer**

(a) The partial fraction expansion method yields

$$H(z) = \frac{1+.5z^{-1}}{(1-.25z^{-1})(1-.75z^{-1})} = \left(1+.5z^{-1}\right)\left(\frac{A_1}{(1-.25z^{-1})} + \frac{A_2}{(1-.75z^{-1})}\right)$$

$$= \left(1+.5z^{-1}\right)\left(\frac{1}{(1-.75/.25)(1-.25z^{-1})} + \frac{1}{(1-.25/.75)(1-.75z^{-1})}\right)$$

$$= \left(1+.5z^{-1}\right)\left(\frac{-.5}{(1-.25z^{-1})} + \frac{1.5}{(1-.75z^{-1})}\right)$$

*Corrected* :

$$\Rightarrow h(n) = -.5(.25)^n u(n) + 1.5(.75)^n u(n) - (.25)^n u(n-1) + (.75)^n u(n-1)$$

*Alternatively* :

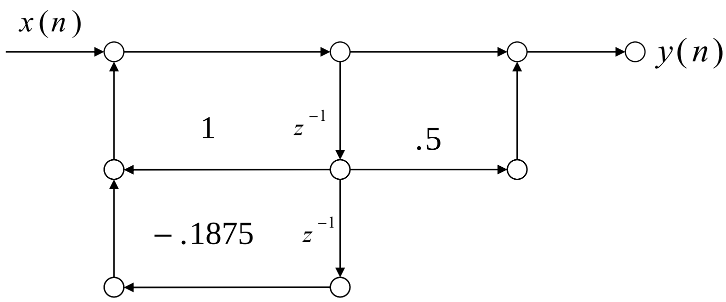
$$H(z) = \frac{1+.5z^{-1}}{(1-.25z^{-1})(1-.75z^{-1})} = \frac{A_1}{(1-.25z^{-1})} + \frac{A_2}{(1-.75z^{-1})}$$

$$A_1 = -1.5$$

$$A_2 = 2.5$$

$$\Rightarrow h(n) = -1.5(.25)^n u(n) + 2.5(.75)^n u(n)$$

(b)




**(20 points) Problem 3:** The impulse response of a channel to be equalized is given by

$h(n) = (-.5)^n [u(n) - u(n-10)]$

where $u(n)$ is the unit step function.

(a) Let $Y(z) = H(z)X(z)$ be the output of the above channel where $H(z) = Z(h(n))$, and $X(z)$ is the z-transform of the input signal $x(n)$. Find the transfer function of an equalizer $H_e(z)$ such that $X(z) = H_e(z)Y(z)$.

(b) What are the locations of the pole(s) and zero(s) of $H(z)$? Are the channel $H(z)$ and equalizer $H_e(z)$ minimum phase? Explain your answer.

**Answer:**

(a)

$$H(z) = \sum_{n=0}^{9} (-.5)^n z^{-n} = \frac{1 - .5^{10} z^{-10}}{1 + .5z^{-1}}$$

$$H_e(z) = \frac{1 + .5z^{-1}}{1 - .5^{10} z^{-10}}$$

(b) There is a single pole of $H(z)$ at $p_1 = -.5$. The locations of the zeros satisfy

$$z_k = .5\exp(j\theta_k),\ k = 0,1,\ldots 9, \qquad .5^{10}(.5\exp(j\theta_k))^{-10} = 1$$

$$\Rightarrow \exp(-j10\,\theta_k) = 1$$

$$\Rightarrow 10\,\theta_k = k2\pi, k = 0,1,\ldots 9$$

$$z_k = .5\exp(jk\pi/5),\ k = 0,1,\ldots 9$$

The pole and 10 zeros all lie within the unit circle, hence both the channel and equalizer are minimum phase.

*Corrected:* The k=5 zero cancels the pole. So H(z) actually has no poles and 9 zeros:

$$z_k = .5\exp(jk\pi/5),\ \ k = 0,1,2,3,4$$

$$z_k = .5\exp(j(k+1)\pi/5),\ \ k = 5,6,7,8$$

**(20 points) Problem 4:** A double notch digital filter $H(z)$ is to be designed using the Bilinear Transform method. The analog filter prototype $H_c(s)$ to be used is given as follows, where $\alpha, \beta$ are constants to be determined using frequency prewarping.

$$H_c(s) = \frac{(s^2+\alpha^2)(s^2+\beta^2)}{(s+1)^4}$$

(a) The digital double notch filter $H(z)$ response satisfies $\left|H(e^{j\omega})\right| = 0$ at the frequencies $\omega = \pi/4, \pi/2$. What are the appropriate values of the constants $\alpha, \beta$ in $H_c(s)$?

(b) Using the values of $\alpha, \beta$ from part (a), find $H(z)$ using the Bilinear Transform method as a ratio of products of polynomials in $z^{-1}$. Express H(z) in a reasonably compact form, but you do not have to multiply out all the polynomials in the numerator and denominator.

**Answer**:

(a) $\alpha = 2\tan\left(\frac{\pi/4}{2}\right) = .8284$ , $\beta = 2\tan\left(\frac{\pi/2}{2}\right) = 2$

(b)

$$H(z) = H_c(s)\Big|_{s = 2\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{\left(4\frac{(1-z^{-1})^2}{(1+z^{-1})^2} + \alpha^2\right)\left(4\frac{(1-z^{-1})^2}{(1+z^{-1})^2} + \beta^2\right)}{\left(2\frac{(1-z^{-1})}{(1+z^{-1})} + 1\right)^4}$$

$$\Rightarrow H(z) = \frac{\left(4(1-z^{-1})^2 + .6862\ (1+z^{-1})^2\right)\left(4(1-z^{-1})^2 + 4(1+z^{-1})^2\right)}{\left(2(1-z^{-1}) + (1+z^{-1})\right)^4}$$

**(20 points) Problem 5:** A six-point sequence $x(n)$ is given by

$$x(n) = 10\,\delta(n) + 8\delta(n-1) + 6\delta(n-2) + 4\delta(n-3) + 2\delta(n-4) + \delta(n-5)$$

(a) Compute the **eight-point** DFT sequence $X(k)$ of $x(n)$ using zero-padding. Express your answer in terms $W_8 = \exp(-j2\pi/8)$ .

(b) Use any method to evaluate the circular convolution of the zero-padded length 8 sequence x(n) with the impulse response $h(n) = \delta(n) - \delta(n-1)$ .

**Answer**

(a) $X(k) = \sum_{n=0}^{8} x(n) W_8^{kn} = 10\,W_8^{\;0} + 8W_8^{\;k} + 6W_8^{\;2k} + 4W_8^{\;3k} + 2W_8^{\;k4} + W_8^{\;5k}$

(b) ***Corrected***:

$Y(k) = X(k)H(k)$

$H(k) = \sum_{n=0}^{1} h(n) W_8^{nk} = W_8^{\;0} - W_8^{\;k}$

$\Rightarrow Y(k) = X(k)H(k) = X(k)\left(W_8^{\;0} - W_8^{\;k}\right)$

$IFFT\;(W_8^{\;k} X(k)) = x((n-1)_8)$

$\Rightarrow\; y(n) = x(n) - x((n-1)_8)$

$= (10\,\delta(n) + 8\delta(n-1) + 6\delta(n-2) + 4\delta(n-3) + 2\delta(n-4) + \delta(n-5))$

$-\,(10\,\delta(n-1) + 8\delta(n-2) + 6\delta(n-3) + 4\delta(n-4) + 2\delta(n-5) + \delta(n-6))$

$\Rightarrow\; y(n) = 10\,\delta(n) - 2\delta(n-1) - 2\delta(n-2) - 2\delta(n-3) - 2\delta(n-4) - \delta(n-5) - \delta(n-6)$