

Sampling and Quantization

Lecture #5
January 21, 2002

Sampling and Quantization

- Spatial Resolution (Sampling)
 - Determines the smallest perceivable image detail.
 - What is the *best* sampling rate?
- Gray-level resolution (Quantization)
 - Smallest discernible change in the gray level value.
 - Is there an optimal quantizer?

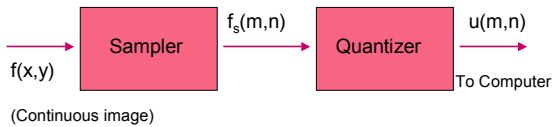
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Image sampling and quantization

In 1-D

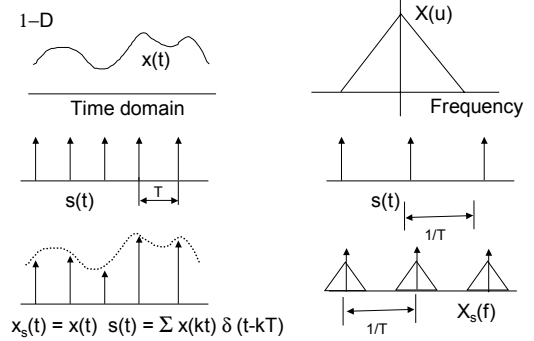


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1-D



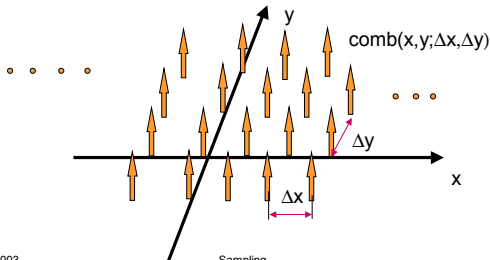
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2-D: Comb function

$$\text{Comb}(x, y; \Delta x, \Delta y) \equiv \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$



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Sampled Image

$$f_s(x, y) = f(x, y) \text{comb}(x, y; \Delta x, \Delta y)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m\Delta x, n\Delta y) \delta(x - m\Delta x, y - n\Delta y)$$

$$\text{comb}(x, y; \Delta x, \Delta y) \xrightarrow{\mathcal{F}} \text{COMB}(u, v) =$$

$$\frac{1}{\Delta x \Delta y} \text{comb}\left(u, v; \frac{1}{\Delta x}, \frac{1}{\Delta y}\right)$$

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Sampled Spectrum

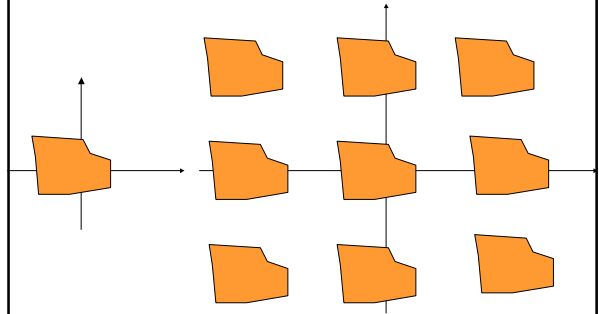
$$\begin{aligned}
 F_s(u, v) &= F(u, v) * \text{COMB}(u, v) \\
 &= \frac{1}{\Delta x \Delta y} \sum_{k, l=-\infty}^{\infty} F(u, v) * \delta\left(u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y}\right) \\
 &= \frac{1}{\Delta x \Delta y} \sum_{k, l=-\infty}^{\infty} F\left(u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y}\right)
 \end{aligned}$$

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Sampled Spectrum: Example



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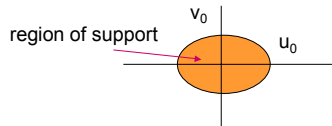
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Bandlimited Images

A function $f(x, y)$ is said to be band limited if the Fourier transform

$$F(u, v) = 0 \quad \text{for } |u| > u_0, |v| > v_0$$

u_0, v_0 \Rightarrow Band width of the image in the x- and y- directions



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Foldover Frequencies

Sampling frequencies:

Let u_s and v_s be the sampling frequencies

$$\text{Then } u_s > 2u_0; v_s > 2v_0$$

$$\text{or } \Delta x < 1/2u_0; \Delta y < 1/2v_0$$

Frequencies above half the sampling frequencies are called fold over frequencies.

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Sampling Theorem

A band limited image $f(x, y)$ with $F(u, v)$ as its Fourier transform; and $F(u, v) = 0$ $|u| > u_0$ $|v| > v_0$; and sampled uniformly on a rectangular grid with spacing Δx and Δy , can be recovered without error from the sample values $f(m \Delta x, n \Delta y)$ provided the sampling rate is greater than the nyquist rate.

$$\text{i.e. } 1/\Delta x = u_s > 2u_0, \quad 1/\Delta y = v_s > 2v_0$$

The reconstructed image is given by the interpolation formula:

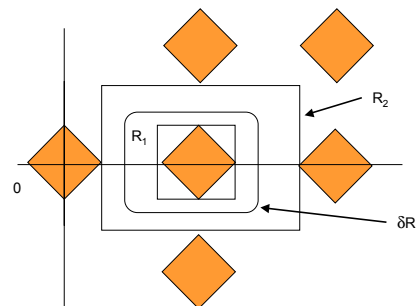
$$f(x, y) = \sum_{m, n=-\infty}^{\infty} f(m \Delta x, n \Delta y) \frac{\sin(x u_s - m)\pi}{(x u_s - m)\pi} \frac{\sin(y v_s - n)\pi}{(y v_s - n)\pi}$$

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Reconstruction



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Reconstruction via LPF

$F(u,v)$ can be recovered by a LPF with

$$H(u,v) = \begin{cases} \Delta x \Delta y & (u,v) \in R \\ 0 & \text{Other wise} \end{cases}$$

R is any region whose boundary ∂R is contained within the annular ring between the rectangles R_1 and R_2 in the figure. Reconstructed signal is

$$\tilde{F}(u,v) = H(u,v) F_s(u,v) = F(u,v)$$

$$f(x,y) = \mathcal{Z}^{-1}[F(u,v)]$$

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Aliasing

Note: If u_s and v_s are below the Nyquist rate, the periodic replications will overlap, resulting in a distorted spectrum.

This overlapping of successive periods of the spectrum causes the foldover frequencies in the original image to appear as frequencies below $u_s/2, v_s/2$ in the sampled image. This is called aliasing.

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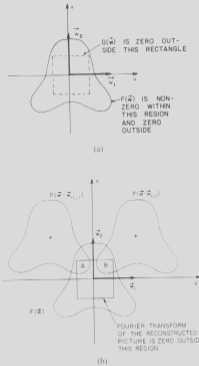


Fig. 9 (a) The sampling lattice in Experiment 1 is used to sample a picture whose Fourier transform is nonzero over a larger region than would lead to artifact-free reconstruction. (b) Those of the terms in Eq. (11) are pictorially illustrated here for $F(u,v)$ shown in Fig. 9a. These three terms correspond to (n,m) equal to $(0,0)$, $(1,-1)$, and $(1,1)$.

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Example

$$f(x,y) = 2 \cos(2\pi(3x+4y))$$

$$F(u,v) = \delta(u-3, v-4) + \delta(u+3, v+4)$$

$$\Rightarrow u_0 = 3, \quad v_0 = 4$$

$$\text{Let } \Delta x = \Delta y = 0.2, \Rightarrow u_s = v_s = \frac{1}{0.2} = 5 < 2u_0, < 2v_0$$

there will be aliasing.

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Example:(contd.)

$$F_s(u,v) = 25 \sum_{k,l=-\infty}^{\infty} \sum F(u-ku, v-lv)$$

$$= 25 \sum_{k,l=-\infty}^{\infty} [\delta(u-3-5k, v-4-5l) + \delta(u+3-5k, v+4-5l)]$$

$$\text{Let } H(u,v) = \begin{cases} 1/25 & -2.5 \leq u \leq 2.5, \quad -2.5 \leq v \leq 2.5 \\ 0 & \text{Otherwise} \end{cases}$$

$$\therefore F(u,v) = H(u,v) F_s(u,v) = \delta(u+2, v+1) + \delta(u-2, v-1)$$

$$\therefore \tilde{f}(x,y) = 2 \cos(2\pi(2x+y))$$

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Examples

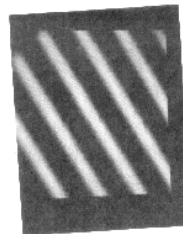


Fig. 6. $\cos(2\pi(6x-4y))$

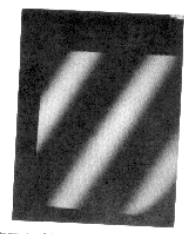


Fig. 7. The reconstructed picture from the samples of the image in Fig. 6. Note the change in frequency and orientation.

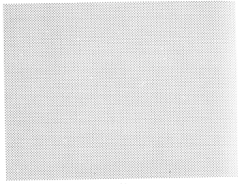
Original and the reconstructed image from samples.

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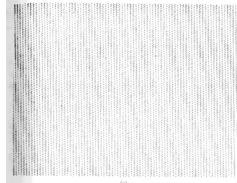
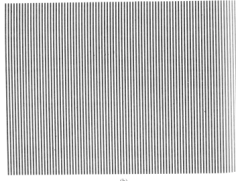
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Another example



Sampling filter



sampled image

Aliasing Problems (real images!)

