

Quantization (Jan 21, 2003)

- Optimal Quantizer
- Uniform Quantizer

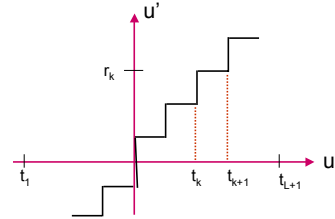
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Image Quantization

u (continuous) $\xrightarrow{\text{Quantizer}}$ $u' \in \{r_1, r_2, r_3, \dots, r_L\}$



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Decision/Reconstruction Levels

$$u \in [t_k, t_{k+1}] \longrightarrow r_k$$

$\{t_k : k=1,2,\dots,L+1\}$ \longleftarrow Transition or decision levels

$r_k \longrightarrow k^{\text{th}}$ reconstruction level

Example: Uniform quantizer $u \in [0,10.0]$

We want $u' \in \{0,1,\dots,255\}$

$t_1 = 0; t_{257} = 10.0$; uniformly spaced, $t_k = (k-1) \cdot 10/256$
 $k = 1,2,\dots,257$

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Example: quantization

$$r_k = t_k + \frac{1}{2} \left(\frac{10}{256} \right) = t_k + \frac{5}{256}$$

Quantization interval

$$q_k = t_k - t_{k-1} = r_k - r_{k-1}$$

= Constant \Rightarrow Uniform quantizer

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MMSE Quantizer

Minimise the mean squared error, MSE = Expected value of $(u-u')^2$ given the number of quantization levels L .

Assume that the density function $p_u(\mathbf{u})$ is known (or can be approximated by a normalised histogram).

Note that for images, u = image intensity. $p_u(\mathbf{u})$ is the image intensity distribution.

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Optimum MSE quantizer

$$E(u) = \text{Expected value of } u = \int_{-\infty}^{\infty} u p_u(u) du$$

$$MSE, \mathcal{E} = E((u-u')^2) = \int_{-\infty}^{\infty} (u-u')^2 p_u(u) du = \int_{t_1}^{t_{L+1}} (u-u')^2 p_u(u) du$$

Since $u' = r_k$ if $u \in [t_k, t_{k+1}]$, we can rewrite this as

$$\mathcal{E} = \sum_{i=1}^L \int_{t_i}^{t_{i+1}} (u-r_i)^2 p_u(u) du$$

Conditions for minimisation of \mathcal{E} are: $\frac{\partial \mathcal{E}}{\partial r_k} = 0$; $\frac{\partial \mathcal{E}}{\partial t_k} = 0$

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MMSE (contd.)

$$\frac{\partial \mathcal{E}}{\partial t_k} = (t_k - r_{k-1})^2 p_u(t_k) - (t_k - r_k)^2 p_u(t_k) = 0$$

$$\text{Now } t_k \leq r_k < t_{k+1} \Rightarrow t_k - r_{k-1} = r_k - t_k \Rightarrow \left[t_k = \frac{(r_k + r_{k-1})}{2} \right] \dots (A)$$

$$\frac{\partial \mathcal{E}}{\partial r_k} = \int_{t_k}^{t_{k+1}} 2(u - r_k)(-1) p_u(u) du = 0$$

$$\Rightarrow r_k = \frac{\int_{t_k}^{t_{k+1}} u p_u(u) du}{\int_{t_k}^{t_{k+1}} p_u(u) du} = E(u | u \in [t_k, t_{k+1}]) \dots (B)$$

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Optimum transition/reconst.

- (1) Optimal transition levels lie halfway between the optimum reconstruction levels.
- (2) Optimum reconstruction levels lie at the center of mass of the probability density in between the transition levels.
- (3) A and B are simultaneous non-linear equations (in general)
 - Closed form solutions normally don't exist → use numerical techniques

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Uniform optimal quantizer

$$\text{Consider } p_u(u) = \begin{cases} \frac{1}{t_{L+1} - t_1} & t_1 \leq u \leq t_{L+1} \\ 0 & \text{Otherwise} \end{cases}$$

$$\text{Then } r_k = \frac{\int_{t_k}^{t_{k+1}} u p_u(u) du}{\int_{t_k}^{t_{k+1}} p_u(u) du} = \frac{1}{t_{L+1} - t_1} \frac{\int_{t_k}^{t_{k+1}} u du}{\int_{t_k}^{t_{k+1}} du} = \frac{\left(\frac{u^2}{2}\right)_{t_k}^{t_{k+1}}}{(u)_{t_k}^{t_{k+1}}} = \frac{(t_{k+1}^2 - t_k^2)}{2(t_{k+1} - t_k)}$$

$$r_k = \frac{1}{2}(t_{k+1} + t_k); \quad t_k = \frac{r_k + r_{k-1}}{2} = \frac{1}{2} \cdot \frac{1}{2} [t_{k+1} + t_k + t_k + t_{k-1}] = \frac{t_{k+1} + t_{k-1}}{2}$$

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Uniform Quantizer

$$t_k - t_{k-1} = t_{k+1} - t_k = \text{Constant } q = \frac{t_{L+1} - t_1}{L}$$

$$t_k = t_{k-1} + q; \quad r_k = t_k + \frac{q}{2}$$

Quantization error $e = (u - u')$ is uniformly distributed over the interval $\left(-\frac{q}{2}, \frac{q}{2}\right)$

$$\text{Mean squared error } E((u - u')^2) = \frac{1}{q} \int_{-\frac{q}{2}}^{\frac{q}{2}} u^2 du = \frac{q^2}{12}$$

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