

**UNIVERSITY of CALIFORNIA**  
**Santa Barbara**  
**Department of Electrical and Computer Engineering**

**ECE 178**  
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**Midterm Exam I**

Date: February 4, 2003.  
 Time: 12:30 – 1:45 PM  
 Max Points: 100

**NAME:** \_\_\_\_\_

2D Fourier Transform:

$$f(x, y) = A \cos 2\pi(ax + by) \Rightarrow F(u, v) = \frac{A}{2} (\delta(u + a, v + b) + \delta(u - a, v - b))$$

MMSE Quantizer:

$$r_k = \frac{\int_{t_k}^{t_{k+1}} u p_U(u) du}{\int_{t_k}^{t_{k+1}} p_U(u) du}, t_k = \frac{r_{k-1} + r_k}{2}$$

Useful Identities

$$2 \cos a \cos b = \cos(a + b) + \cos(a - b)$$

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a^2 - x^2 = (a - x)(a + x)$$

$$a^3 - x^3 = (a - x)(a^2 + ax + x^2)$$

1. (20) Given a discrete sequence  $u(m, n) = (m + n)^2$ , evaluate:

(a) (10)  $u(m, n) \delta(m - 2, n - 2)$

$$\delta(m - 2, n - 2) = \begin{cases} 1 & m = 2, n = 2 \\ 0 & \text{otherwise} \end{cases} \Rightarrow u(m, n) \delta(m - 2, n - 2) = \begin{cases} 1 * u(m, n) & m = 2, n = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$u(m, n) \delta(m - 2, n - 2) = \begin{cases} (m + n)^2 & m = 2, n = 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 16 & m = 2, n = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{u(m, n) \delta(m - 2, n - 2) = 16 \delta(m - 2, n - 2)}$$

(b) (10)  $u(m, n) * \delta(m - 2, n - 2)$

$$\delta(m - 2, n - 2) = \begin{cases} 1 & m = 2, n = 2 \\ 0 & \text{otherwise} \end{cases} (*)$$

$$u(m, n) * \delta(m - 2, n - 2) = \sum_{m'=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} u(m', n') \delta(m - 2 - m', n - 2 - n') = u(m - 2, n - 2) (*)$$

$$u(m, n) * \delta(m - 2, n - 2) = ((m - 2) + (n - 2))^2$$

$$\boxed{u(m, n) * \delta(m - 2, n - 2) = (m + n - 4)^2}$$

2. (20) Determine if the following systems are (i) Linear and (ii) Shift-Invariant.

(a)  $y(m, n) = e^{-|x(m, n)|}$ ,  $y[m, n] = T[x[m, n]] = e^{-|x(m, n)|}$

(i) (5) linear if  $T[ax_1[m, n] + bx_2[m, n]] = ay_1[m, n] + by_2[m, n]$ .

Change variables:  $x'[m, n] = ax_1[m, n] + bx_2[m, n]$

$T[ax_1[m, n] + bx_2[m, n]] = T[x'[m, n]] = e^{-|x'(m, n)|} = e^{-|ax_1[m, n] + bx_2[m, n]|} \neq$

$\neq ae^{-|x_1(m, n)|} + be^{-|x_2(m, n)|} = ay_1[m, n] + by_2[m, n]$ , **system is not linear.**

(ii) (5) shift-invariant if  $T[x[m - m_0, n - n_0]] = y[m - m_0, n - n_0]$ .

Change variables:  $x'[m, n] = x[m - m_0, n - n_0]$

$T[x[m - m_0, n - n_0]] = T[x'[m, n]] = e^{-|x'(m, n)|} = e^{-|x[m - m_0, n - n_0]|} = y[m - m_0, n - n_0]$ ,

**system is shift invariant**

(b) (10)  $y(m, n) = \sum_{k=0}^n x(m-1, k)$

(i) (5) linear if  $T[ax_1[m, n] + bx_2[m, n]] = ay_1[m, n] + by_2[m, n]$ .

Change variables:  $x'[m, n] = ax_1[m, n] + bx_2[m, n]$ ,  $T[ax_1[m, n] + bx_2[m, n]] = T[x'[m, n]]$

$T[x'[m, n]] = \sum_{k=0}^n x'[m-1, k] = \sum_{k=0}^n ax_1[m-1, k] + bx_2[m-1, k] = a \sum_{k=0}^n x_1[m-1, k] + b \sum_{k=0}^n x_2[m-1, k] =$

$= ay_1[m, n] + by_2[m, n]$ , **system is linear**

(ii) (5) shift-invariant if  $T[x[m - m_0, n - n_0]] = y[m - m_0, n - n_0]$ .

Change variables:  $x'[m, n] = x[m - m_0, n - n_0]$ ,  $T[x[m - m_0, n - n_0]] = T[x'[m, n]]$

$T[x'[m, n]] = \sum_{k=0}^n x'[m-1, k] = \sum_{k=0}^m x'[m-1, k] = \sum_{k=0}^m x[m - m_0 - 1, k - n_0] = \sum_{k'=-n_0}^{n-n_0} x[m - m_0 - 1, k'] \neq$

$\neq \sum_{k'=0}^{n-n_0} x[m - m_0 - 1, k'] = y[m - m_0, n - n_0]$ , **system is not shift invariant.**

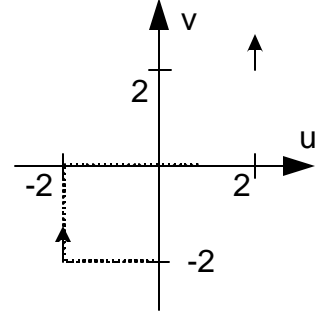
(30) The image  $f(x, y) = 4\cos 4\pi(x + y)$  is sampled at  $\Delta x = 0.25$ ,  $\Delta y = 0.25$ . The reconstructed filter is the ideal low pass filter corresponding to the above sampling rate. What is the reconstructed signal?

$$f(x, y) = 4\cos 2\pi(2x + 2y) \Rightarrow u_0 = 2, v_0 = 2$$

$$F(u, v) = 2(\delta(u + 2, v + 2) + \delta(u - 2, v - 2))$$

$$u_s = \frac{1}{\Delta x} = 4, 2u_0 = 4 \Rightarrow u_s = 2u_0, \text{ aliasing exists.}$$

$$v_s = \frac{1}{\Delta y} = 4, 2v_0 = 4 \Rightarrow v_s = 2v_0, \text{ aliasing exists.}$$

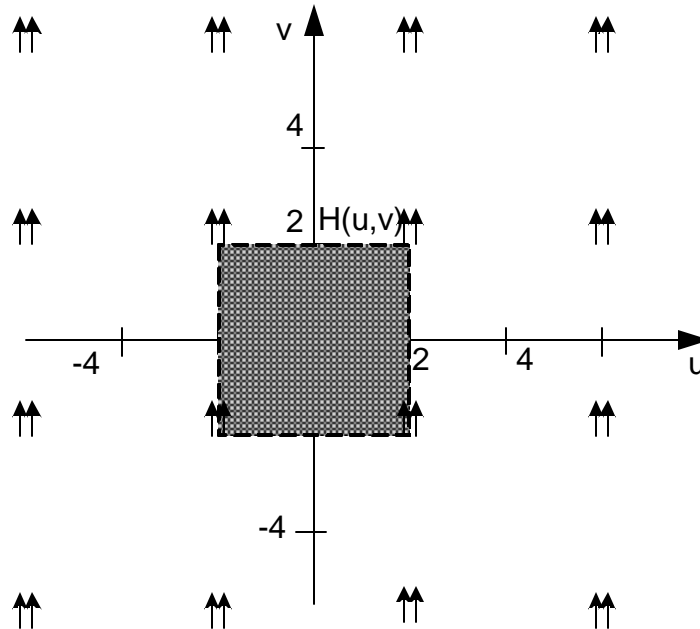


Spectrum of a sampled signal is:

$$F_s(u, v) = 32 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (\delta(u + 2 - 4k, v + 2 - 4l) + \delta(u - 2 - 4k, v - 2 - 4l))$$

The ideal low pass filter  $H(u, v)$  corresponding to the above sampling rate, i.e. :

$$H(u, v) = \begin{cases} 1 & -2 \leq u \leq 2 \\ \frac{1}{16} & -2 \leq v \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

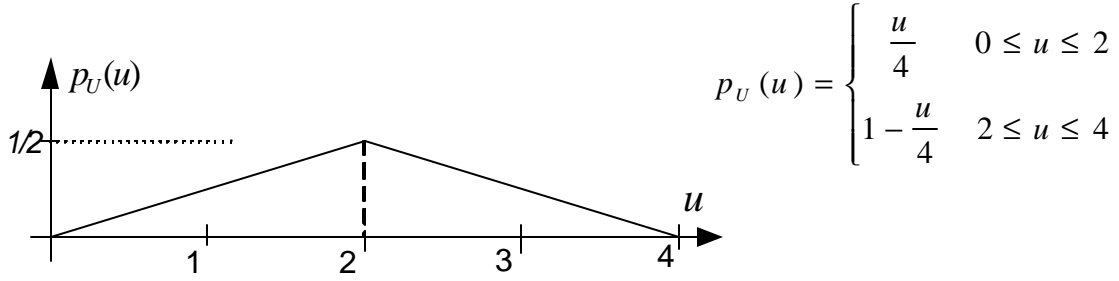


Reconstructed signal is:

$$\hat{F}(u, v) = H(u, v) F_s(u, v) = 0.0625 * 32 * 2 (\delta(u + 2, v + 2) + \delta(u + 2, v - 2) + \delta(u - 2, v + 2) + \delta(u - 2, v - 2))$$

$$\hat{f}(x, y) = 8\cos(4\pi(x + y)) + 8\cos(4\pi(x - y)) \Rightarrow \boxed{\hat{f}(x, y) = 16\cos 4\pi x \cos 4\pi y}$$

4. (30) Design a four level optimal minimum mean squared error quantizer for  $u$  where the signal has the following pdf.



$$\text{-MMSE quantizer: } r_k = \frac{\int_{t_k}^{t_{k+1}} u p_U(u) du}{\int_{t_k}^{t_{k+1}} p_U(u) du}, t_k = \frac{r_{k-1} + r_k}{2}$$

$$\text{-pdf symmetry: } \boxed{t_1 = 0}, \boxed{t_3 = 2}, \boxed{t_5 = 4}, t_5 - t_4 = t_2 - t_1 \Rightarrow t_4 = 4 - t_2 \quad (1)$$

$$- r_1 = \frac{\int_0^{t_2} u \frac{1}{4} u du}{\int_0^{t_2} \frac{1}{4} u du} = \frac{\frac{t_2^3}{3}}{\frac{t_2^2}{2}} \Rightarrow r_1 = \frac{2}{3} t_2, t_2 = \frac{r_1 + r_2}{2} \quad (2)$$

$$r_2 = \frac{\int_{t_2}^{t_3} u p_U(u) du}{\int_{t_2}^{t_3} p_U(u) du} = \frac{\int_{t_2}^2 u \frac{1}{4} u du}{\int_{t_2}^2 \frac{1}{4} u du} = \frac{\frac{2^3 - t_2^3}{3}}{\frac{2^2 - t_2^2}{2}} = \frac{2}{3} \frac{4 + 2t_2 + t_2^2}{2 + t_2} \quad (3)$$

$$(2) \ \& \ (3) \Rightarrow t_2 = \frac{r_1 + r_2}{2} \Rightarrow t_2 = \frac{1}{2} \left( \frac{2}{3} t_2 + \frac{2}{3} \frac{4 + 2t_2 + t_2^2}{2 + t_2} \right) \Rightarrow 3t_2(2 + t_2) = 2t_2^2 + 4t_2 + 4$$

$$\Rightarrow t_2^2 + 2t_2 - 4 = 0 \Rightarrow t_2 = -1 \pm \sqrt{5}. \text{ Since } t_1 = 0 \leq t_2 \leq t_3 = 2 \Rightarrow \boxed{t_2 = \sqrt{5} - 1}.$$

$$(1) \Rightarrow t_4 = 4 - t_2 \Rightarrow \boxed{t_4 = 5 - \sqrt{5}}$$

$$(2) \ r_1 = \frac{2}{3} t_2 \Rightarrow \boxed{r_1 = \frac{2}{3}(\sqrt{5} - 1)} \quad - \ (3) \ r_2 = 2t_2 - r_1 = \frac{4}{3} t_2 \Rightarrow \boxed{r_2 = \frac{4}{3}(\sqrt{5} - 1)}$$

$$\text{- pdf symmetry: } r_3 = 4 - r_2 \Rightarrow \boxed{r_3 = \frac{4}{3}(4 - \sqrt{5})}, \quad r_4 = 4 - r_1 \Rightarrow \boxed{r_4 = \frac{2}{3}(7 - \sqrt{5})}$$

SOLUTION:

$$\boxed{t_1 = 0, t_2 = \sqrt{5} - 1, t_3 = 2, t_4 = 5 - \sqrt{5}, t_5 = 5}$$

$$\boxed{r_1 = \frac{2}{3}(\sqrt{5} - 1), r_2 = \frac{4}{3}(\sqrt{5} - 1), r_3 = \frac{4}{3}(4 - \sqrt{5}), r_4 = \frac{2}{3}(7 - \sqrt{5})}$$