UNIVERSITY of CALIFORNIA
Santa Barbara
Department of Electrical and Computer Engineering

ECE 178 | **Midterm Exam I** | Date: February 4, 2003.
B.S. Manjunath | Time: 12:30 – 1:45 PM
TAs: J. Tešic, M. Zuliani | Max Points: 100

NAME: ____

2D Fourier Transform:

$$f(x,y) = A\cos 2\pi(ax+by) \Rightarrow F(u,v) = \frac{A}{2}(\delta(u+a,v+b)+\delta(u-a,v-b))$$

MMSE Quantizer:

$$r_k = \frac{\int_{t_k}^{t_{k+1}} u p_U(u)du}{\int_{t_k}^{t_{k+1}} p_U(u)du}, t_k = \frac{r_{k-1}+r_k}{2}$$

Useful Identities

$$2\cos a\cos b = \cos(a+b)+\cos(a-b)$$

$$ax^2+bx+c=0 \Rightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$a^2-x^2=(a-x)(a+x)$$

$$a^3-x^3=(a-x)(a^2+ax+x^2)$$

1. (20) Given a discrete sequence $u(m,n)=(m+n)^2$, evaluate:

(a) (10) $u(m,n)\delta(m-2,n-2)$

$$\delta(m-2,n-2) = \begin{cases}1 & m=2,n=2\\0 & otherwise\end{cases} \Rightarrow u(m,n)\delta(m-2,n-2) = \begin{cases}1*u(m,n) & m=2,n=2\\0 & otherwise\end{cases}$$

$$u(m,n)\delta(m-2,n-2) = \begin{cases}(m+n)^2 & m=2,n=2\\0 & otherwise\end{cases} = \begin{cases}16 & m=2,n=2\\0 & otherwise\end{cases}$$

$$\boxed{u(m,n)\delta(m-2,n-2) = 16\delta(m-2,n-2)}$$

(b) (10) $u(m,n)*\delta(m-2,n-2)$

$$\delta(m-2,n-2) = \begin{cases}1 & m=2,n=2\\0 & otherwise\end{cases} (*)$$

$$u(m,n)*\delta(m-2,n-2) = \sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} u(m',n')\delta(m-2-m',n-2-n') \overset{(*)}{=} u(m-2,n-2)$$

$$u(m,n)*\delta(m-2,n-2) = ((m-2)+(n-2))^2$$

$$\boxed{u(m,n)*\delta(m-2,n-2) = (m+n-4)^2}$$

2. (20) Determine if the following systems are (i) Linear and (ii) Shift-Invariant.

(a) $y(m,n) = e^{-|x(m,n)|}$, $y[m,n] = T[x[m,n]] = e^{-|x(m,n)|}$

(i) (5) linear if $T[ax_1[m,n]+bx_2[m,n]] = ay_1[m,n]+by_2[m,n]$.

Change variables: $x'[m,n] = ax_1[m,n]+bx_2[m,n]$

$T[ax_1[m,n]+bx_2[m,n]] = T[x'[m,n]] = e^{-|x'(m,n)|} = e^{-|ax_1[m,n]+bx_2[m,n]|} \neq$

$\neq ae^{-|x_1(m,n)|} + be^{-|x_2(m,n)|} = ay_1[m,n]+by_2[m,n]$, **system is not linear.**

(ii) (5) shift-invariant if $T[x[m-m_0, n-n_0]] = y[m-m_0, n-n_0]$.

Change variables: $x'[m,n] = x[m-m_0, n-n_0]$

$T[x[m-m_0,n-n_0]] = T[x'[m,n]] = e^{-|x'(m,n)|} = e^{-|x[m-m_0,n-n_0]|} = y[m-m_0, n-n_0]$,

system is shift invariant

(b) (10) $y(m,n) = \sum_{k=0}^{n} x(m-1,k)$

(i) (5) linear if $T[ax_1[m,n]+bx_2[m,n]] = ay_1[m,n]+by_2[m,n]$.

Change variables: $x'[m,n] = ax_1[m,n]+bx_2[m,n]$, $T[ax_1[m,n]+bx_2[m,n]] = T[x'[m,n]]$

$$T[x'[m,n]] = \sum_{k=0}^{n} x'[m-1,k] = \sum_{k=0}^{n} ax_1[m-1,k]+bx_2[m-1,k] = a\sum_{k=0}^{n} x_1[m-1,k] + b\sum_{k=0}^{n} x_2[m-1,k] =$$

$= ay_1[m,n]+by_2[m,n]$, **system is linear**

(ii) (5) shift-invariant if $T[x[m-m_0, n-n_0]] = y[m-m_0, n-n_0]$.

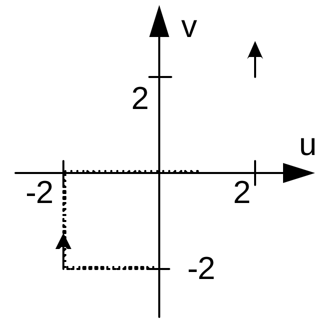
Change variables: $x'[m,n] = x[m-m_0, n-n_0]$, $T[x[m-m_0,n-n_0]] = T[x'[m,n]]$

$$T[x'[m,n]] = \sum_{k=0}^{n} x'[m-1,k] = \sum_{k=0}^{m} x'[m-1,k] = \sum_{k=0}^{m} x[m-m_0-1, k-n_0] = \sum_{k'=-n_0}^{n-n_0} x[m-m_0-1,k'] \neq$$

$\neq \sum_{k'=0}^{n-n_0} x[m-m_0-1,k'] = y[m-m_0, n-n_0]$, **system is not shift invariant.**

(30) The image $f(x,y) = 4\cos 4\pi(x+y)$ is sampled at $\Delta x = 0.25$, $\Delta y = 0.25$. The reconstructed filter is the ideal low pass filter corresponding to the above sampling rate. What is the reconstructed signal?

$$f(x,y) = 4\cos 2\pi(2x+2y) \Rightarrow u_0 = 2, v_0 = 2$$
$$F(u,v) = 2(\delta(u+2,v+2)+\delta(u-2,v-2))$$
$$u_s = \frac{1}{\Delta x} = 4, 2u_0 = 4 \Rightarrow u_s = 2u_0\text{, aliasing exists.}$$
$$v_s = \frac{1}{\Delta y} = 4, 2v_0 = 4 \Rightarrow v_s = 2v_0\text{, aliasing exists.}$$

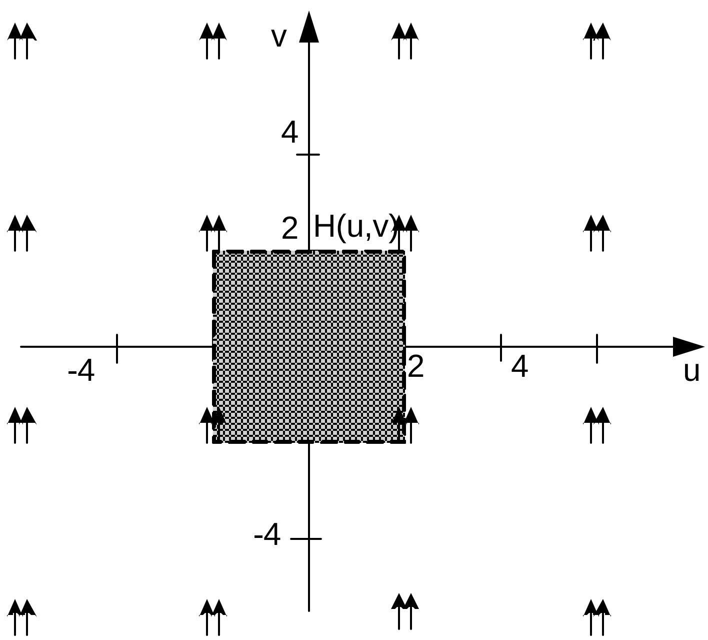


Spectrum of a sampled signal is:

$$F_s(u,v) = 32\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(\delta(u+2-4k, v+1-4l) + \delta(u-2-4k, v-1-4l))$$

The ideal low pass filter *H(u,v)* corresponding to the above sampling rate, i.e. :

$$H(u,v) = \begin{cases} \frac{1}{16}, & \begin{matrix} -2 \le u \le 2 \\ -2 \le v \le 2 \end{matrix} \\ 0, & \text{otherwise} \end{cases}$$

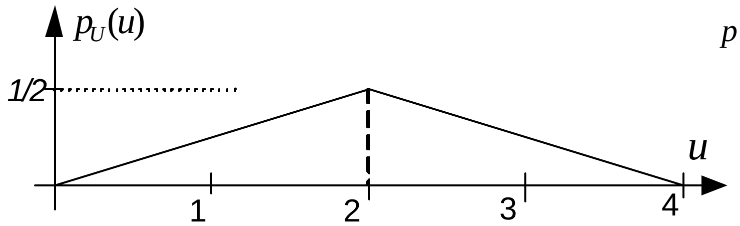


Reconstructed signal is:

$$\hat{F}(u,v) = H(u,v)F_s(u,v) = 0.0625*32*2(\delta(u+2,v+2)+\delta(u+2,v-2)+\delta(u-2,v+2)+\delta(u-2,v-2))$$

$$\hat{f}(x,y) = 8\cos(4\pi(x+y)) + 8\cos(4\pi(x-y)) \Rightarrow \boxed{\hat{f}(x,y) = 16\cos 4\pi x \cos 4\pi y}$$

4. (30) Design a four level optimal minimum mean squared error quantizer for u where the signal has the following pdf.



$$p_U(u) = \begin{cases} \frac{u}{4} & 0 \le u \le 2 \\ 1 - \frac{u}{4} & 2 \le u \le 4 \end{cases}$$

-MMSE quantizer: $r_k = \dfrac{\int_{t_k}^{t_{k+1}} u p_U(u) du}{\int_{t_k}^{t_{k+1}} p_U(u) du}, t_k = \dfrac{r_{k-1} + r_k}{2}$

-pdf symmetry: $\boxed{t_1 = 0}, \boxed{t_3 = 2}, \boxed{t_5 = 4}, t_5 - t_4 = t_2 - t_1 \Rightarrow t_4 = 4 - t_2$ (1)

$$- \; r_1 = \frac{\int_0^{t_2} u \frac{1}{4} u du}{\int_{t_k}^{t2} \frac{1}{4} u du} = \frac{\frac{t_2^3}{3}}{\frac{t_2^2}{2}} \Rightarrow r_1 = \frac{2}{3} t_2, t_2 = \frac{r_1 + r_2}{2} \; (2)$$

$$r_2 = \frac{\int_{t_2}^{t_3} u p_U(u) du}{\int_{t_2}^{t_3} p_U(u) du} = \frac{\int_{t_2}^{2} u \frac{1}{4} u du}{\int_{t_2}^{t_2} \frac{1}{4} u du} = \frac{\frac{2^3 - t_2^3}{3}}{\frac{2^2 - t_2^{\;2}}{2}} = \frac{2}{3} \frac{4 + 2t_2 + t_2^2}{2 + t_2} \; (3)$$

(2) & (3) $\Rightarrow t_2 = \dfrac{r_1 + r_2}{2} \Rightarrow t_2 = \dfrac{1}{2}(\dfrac{2}{3} t_2 + \dfrac{2}{3} \dfrac{4 + 2t_2 + t_2^2}{2 + t_2}) \Rightarrow 3t_2(2 + t_2) = 2t_2^2 + 4t_2 + 4$

$\Rightarrow t_2^2 + 2t_2 - 4 = 0 \Rightarrow t_2 = -1 \pm \sqrt{5}$. Since $t_1 = 0 \le t_2 \le t_3 = 2 \Rightarrow \boxed{t_2 = \sqrt{5} - 1}$.

(1) $\Rightarrow t_4 = 4 - t_2 \Rightarrow \boxed{t_4 = 5 - \sqrt{5}}$

(2) $r_1 = \dfrac{2}{3} t_2 \Rightarrow \boxed{r_1 = \dfrac{2}{3}(\sqrt{5} - 1)}$ - (3) $r_2 = 2t_2 - r_1 = \dfrac{4}{3} t_2 \Rightarrow \boxed{r_2 = \dfrac{4}{3}(\sqrt{5} - 1)}$

- pdf symmetry: $r_3 = 4 - r_2 \Rightarrow \boxed{r_3 = \dfrac{4}{3}(4 - \sqrt{5})}$, $r_4 = 4 - r_1 \Rightarrow \boxed{r_4 = \dfrac{2}{3}(7 - \sqrt{5})}$

SOLUTION:

$$\boxed{\begin{array}{l} t_1 = 0, t_2 = \sqrt{5} - 1, t_3 = 2, t_4 = 5 - \sqrt{5}, t_5 = 5 \\ r_1 = \frac{2}{3}(\sqrt{5} - 1), r_2 = \frac{4}{3}(\sqrt{5} - 1), r_3 = \frac{4}{3}(4 - \sqrt{5}), r_4 = \frac{2}{3}(7 - \sqrt{5}) \end{array}}$$