Sampling and Quantization

Lecture #5

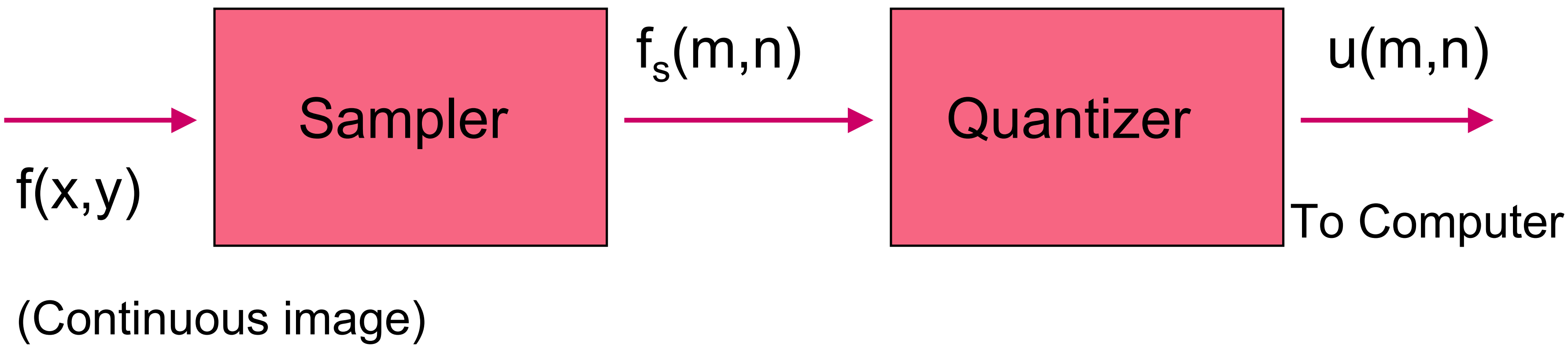
January 21, 2002

Sampling and Quantization

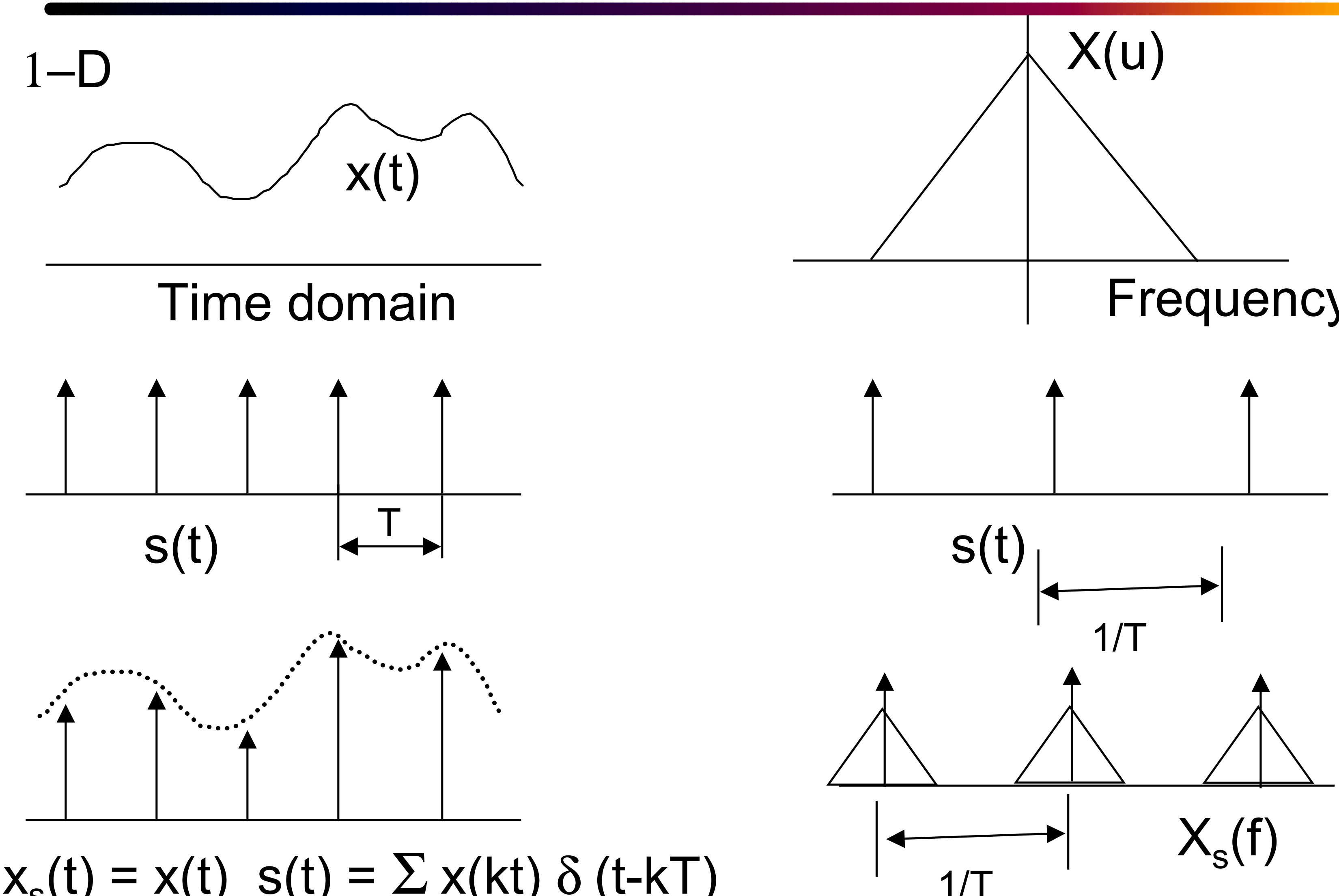
- Spatial Resolution (Sampling)
  - Determines the smallest perceivable image detail.
  - What is the *best* sampling rate?
- Gray-level resolution (Quantization)
  - Smallest discernible change in the gray level value.
  - Is there an optimal quantizer?

Image sampling and quantization

In 1-D



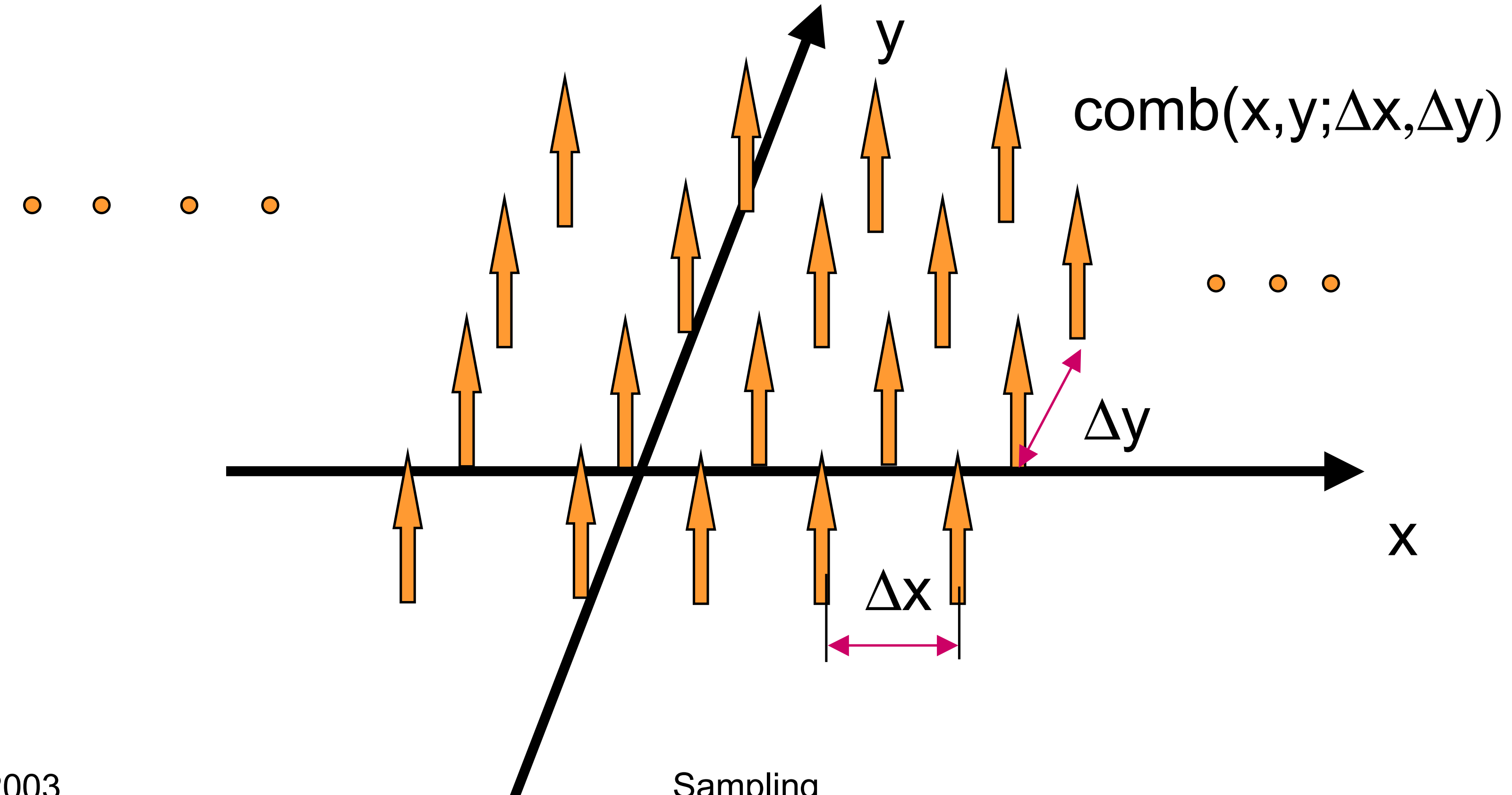
1-D



$x_s(t) = x(t)\ \ s(t) = \Sigma\, x(kt)\, \delta\, (t\text{-}kT)$

2-D: Comb function

$$\mathrm{Comb}(x,y;\Delta x,\Delta y)\cong\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x,y-n\Delta y)$$



Sampled Image

$$f_s(x,y) = f(x,y)\ \text{comb}\ (x,y\ ;\ \Delta x, \Delta y)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m\Delta x, n\Delta y)\ \delta(x - m\Delta x, y - n\Delta y)$$

$$\text{comb}(x,y;\Delta x,\Delta y) \xleftrightarrow{\ \Im\ } \text{COMB}\ (u,v) =$$

$$\frac{1}{\Delta x \Delta y}\text{comb}(u,v\ ;\ \frac{1}{\Delta x},\frac{1}{\Delta y})$$

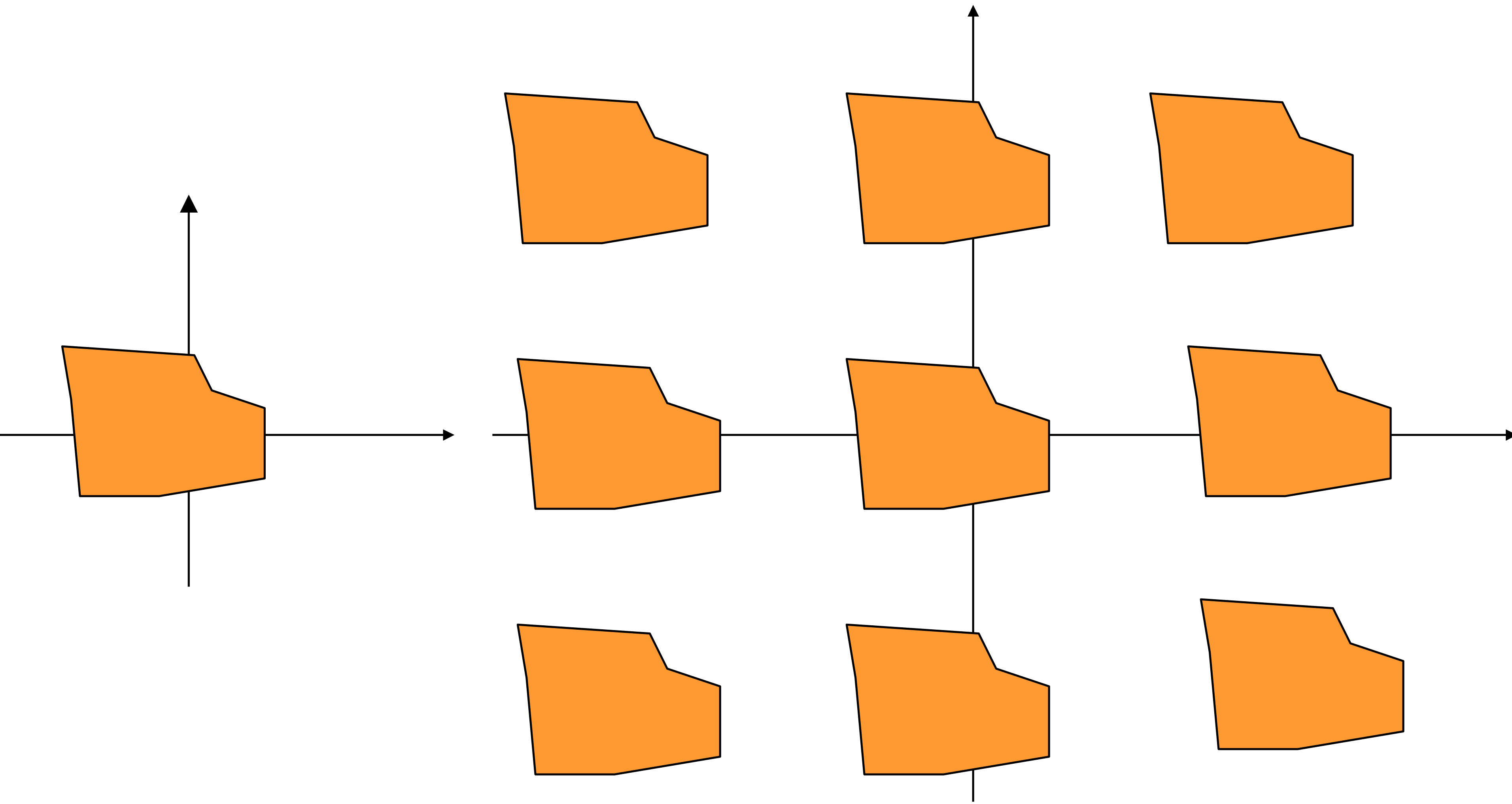
Sampled Spectrum

$$F_s(u,v) = F(u,v) * \mathrm{COMB}(u,v)$$

$$= \frac{1}{\Delta x \Delta y} \sum_{k,l=-\infty}^{\infty} \sum F(u,v) * \delta\left(u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y}\right)$$

$$= \frac{1}{\Delta x \Delta y} \sum_{k,l=-\infty}^{\infty} \sum F\left(u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y}\right)$$

Sampled Spectrum: Example

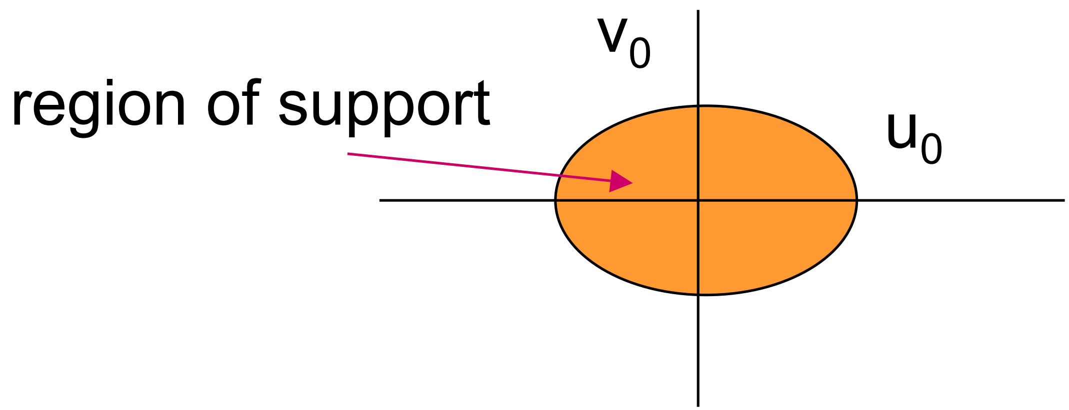


Bandlimited Images

A function f(x,y) is said to be band limited if the Fourier transform

$F(u,v) = 0$ for $|u| > u_0, \ |v| > v_0$

u_0, v_0 → Band width of the image in the x- and y- directions



Foldover Frequencies

Sampling frequencies:
Let u_s and v_s be the sampling frequencies

Then $u_s > 2u_0$; $v_s > 2v_0$

or $\Delta x < 1/2u_0$; $\Delta y < 1/2v_0$

Frequencies above half the sampling frequencies are called fold over frequencies.

Sampling Theorem

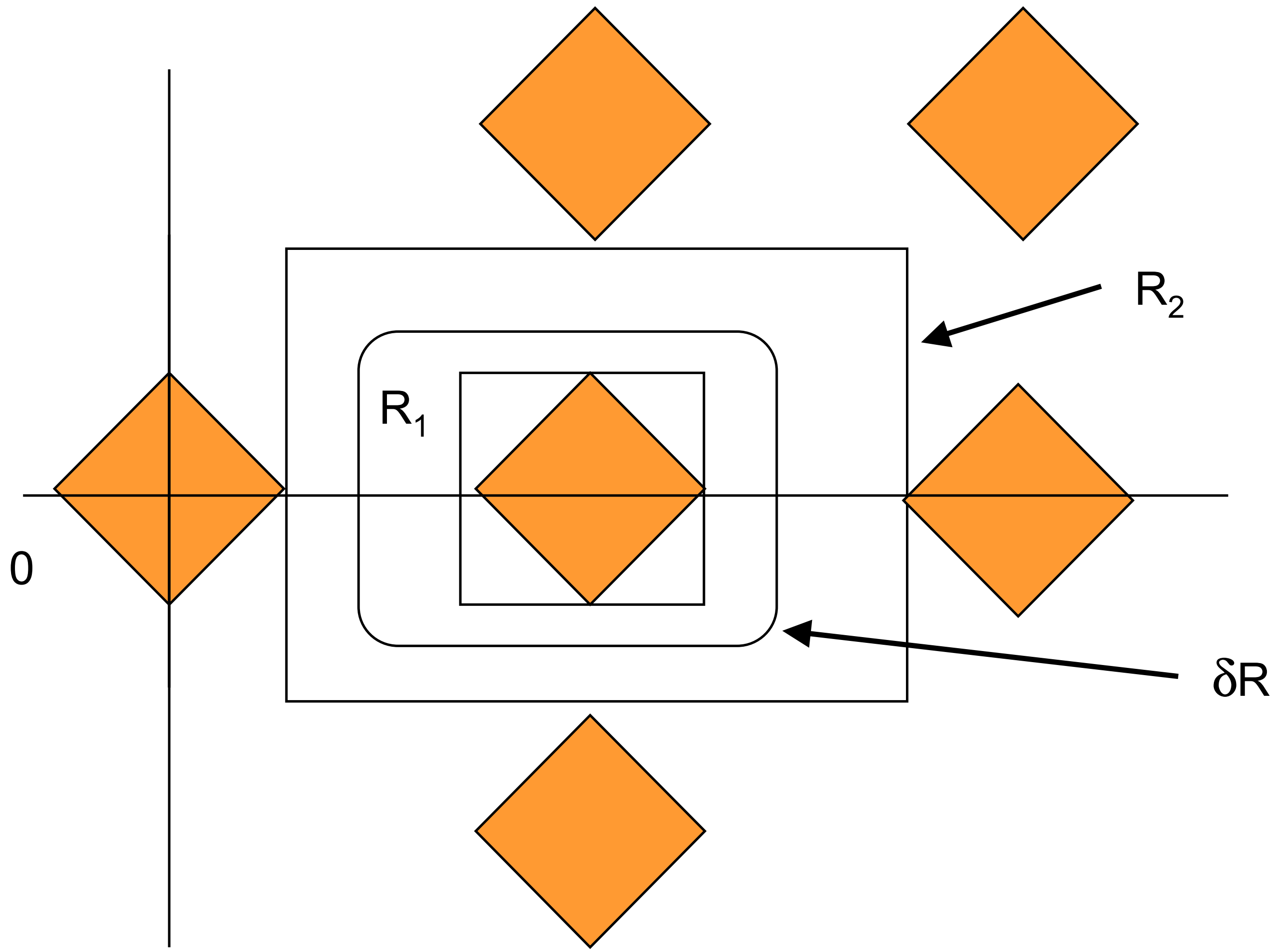
A band limited image f(x,y) with F(u,v) as its Fourier transform; and F(u,v) = 0 $|u| > u_0$ $|v| > v_0$; and sampled uniformly on a rectangular grid with spacing Δx and Δy, can be recovered without error from the sample values $f(m\,\Delta x, n\,\Delta y)$ provided the sampling rate is greater than the nyquist rate.

i.e $1/\Delta x = u_s > 2u_0$, $1/\Delta y = v_s > 2v_0$

The reconstructed image is given by the interpolation formula:

$$f(x,y) = \sum_{m,n=-\infty}^{\infty}\sum f(m\,\Delta x, n\,\Delta y)\ \frac{\sin(x u_s - m)\pi}{(x u_s - m)\pi}\ \frac{\sin(y v_s - n)\pi}{(y v_s - n)\pi}$$

Reconstruction



Reconstruction via LPF

F(u,v) can be recovered by a LPF with

$$H(u,v) = \begin{cases} \Delta x \, \Delta y & (u,v) \in R \\ 0 & \text{Other wise} \end{cases}$$

R is any region whose boundary ∂R is contained within the annular ring between the rectangles R_1 and R_2 in the figure. Reconstructed signal is

$$\tilde{F}(u,v) = H(u,v)\, F_s(u,v) = F(u,v)$$

$$f(x,y) = \Im^{-1}[F(u,v)]$$

Aliasing

Note: If u_s and v_s are below the Nyquist rate, the periodic replications will overlap, resulting in a distorted spectrum.

This overlapping of successive periods of the spectrum causes the foldover frequencies in the original image to appear as frequencies below $u_s/2$, $v_s/2$ in the sampled image. This is called aliasing.

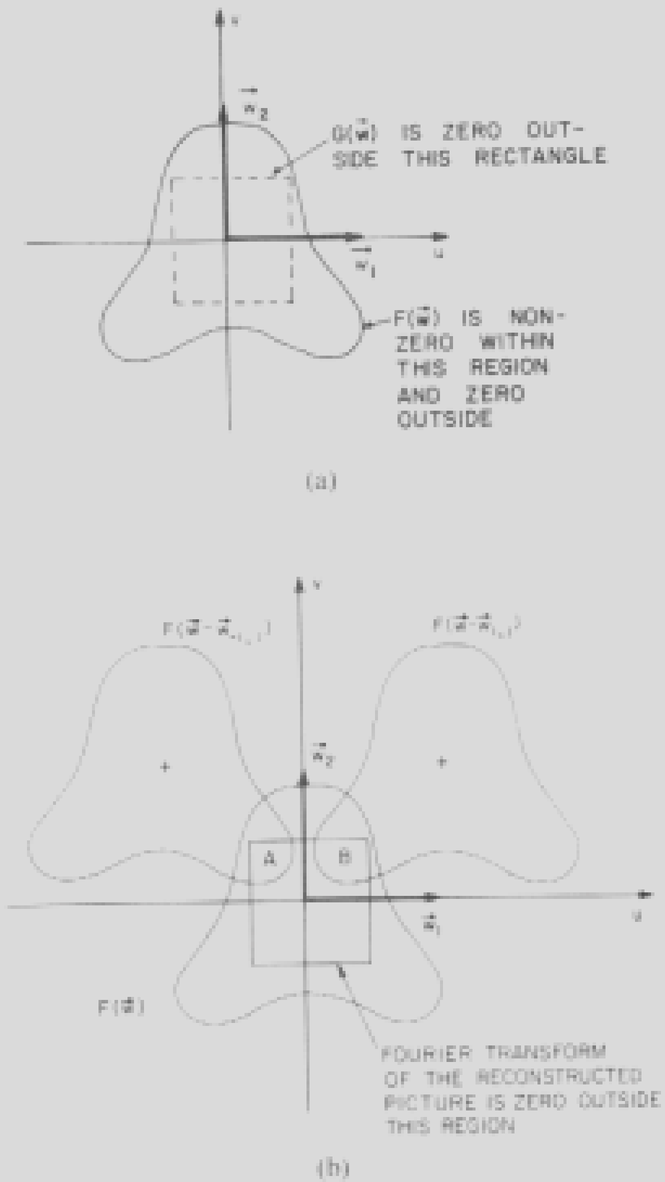


Fig. 9 (a) The sampling lattice in Experiment 1 is used to sample a picture whose Fourier transform is nonzero over a larger region than would lead to error-free reconstruction. (b) Three of the terms in Eq. (31) are pictorially illustrated here for $F(\vec{w})$ shown in Fig. 9a. These three terms correspond to (m, n) equal to (0, 0), (1, −1), and (1, 1).

Example

$$f(x,y) = 2\cos(2\pi(3x+4y))$$

$$F(u,v) = \delta(u-3, v-4) + \delta(u+3, v+4)$$

$$\Rightarrow u_0 = 3, \quad v_0 = 4$$

$$\text{Let} \quad \Delta x = \Delta y = 0.2, \Rightarrow u_s = v_s = \frac{1}{0.2} = 5 \quad < 2u_0, < 2v_0$$

there will be aliasing.

Example:(contd.)

$$F_s(u,v) = 25 \sum_{k,l=-\infty}^{\infty}\sum F(u - ku_s, v - lv_s)$$

$$= 25 \sum_{k,l=-\infty}^{\infty}\sum [\delta(u-3-5k, v-4-5l) + \delta(u+3-5k, v+4-5l)]$$

Let $$H(u,v) = \begin{cases} 1/25 & -2.5 \le u \le 2.5, \quad -2.5 \le u \le 2.5 \\ 0 & \text{Otherwise} \end{cases}$$

$$\therefore F(u,v) = H(u,v)\, F_s(u,v)$$

$$= \delta(u+2, v+1) + \delta(u-2, v-1)$$

$$\therefore \tilde{f}(x,y) = 2\cos(2\pi\,(2x+y))$$

Examples

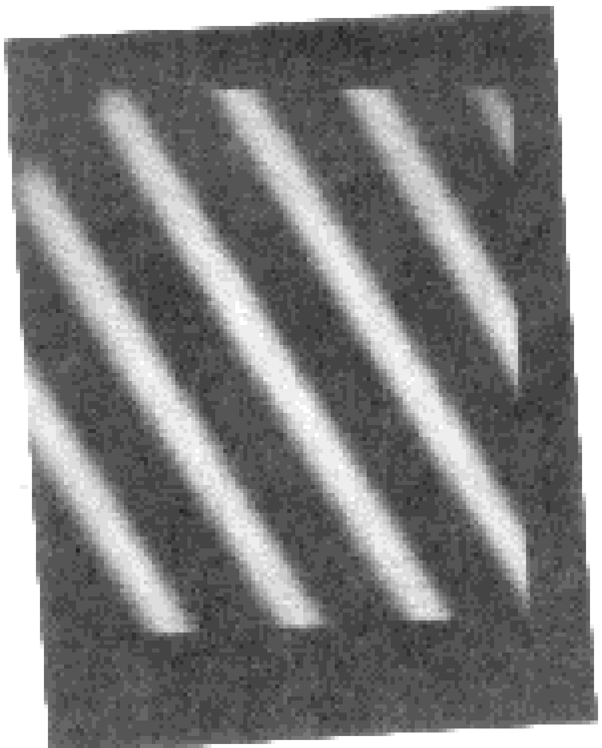
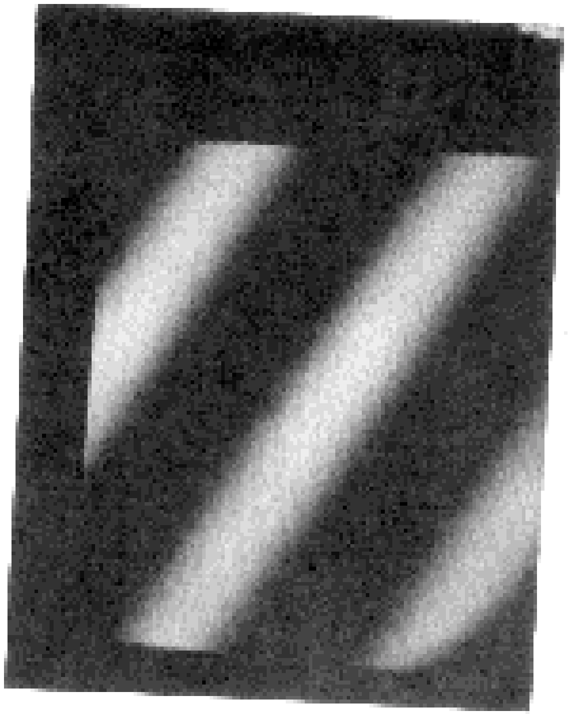


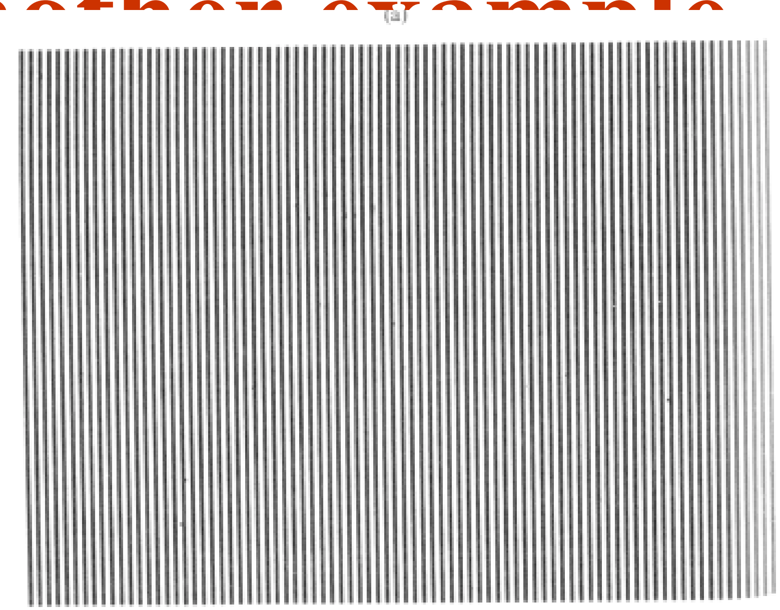
Fig. [illegible]



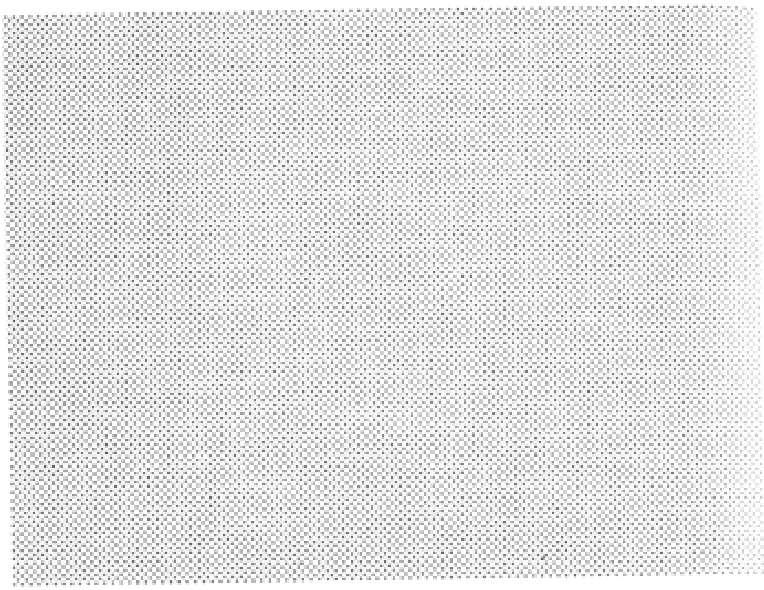
[illegible] The reconstructed picture from the samples of the picture in Fig. 5. Note the change in frequency and orientation.

Original and the reconstructed image from samples.

Another example

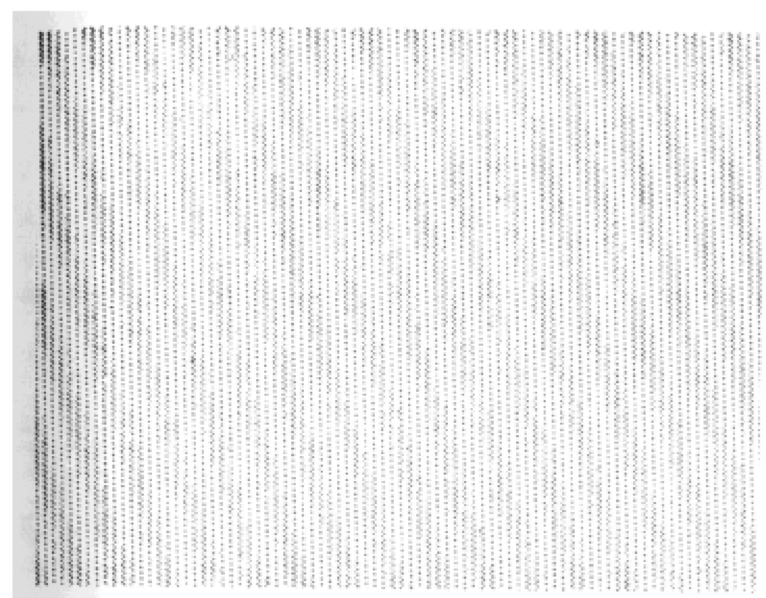


(b)

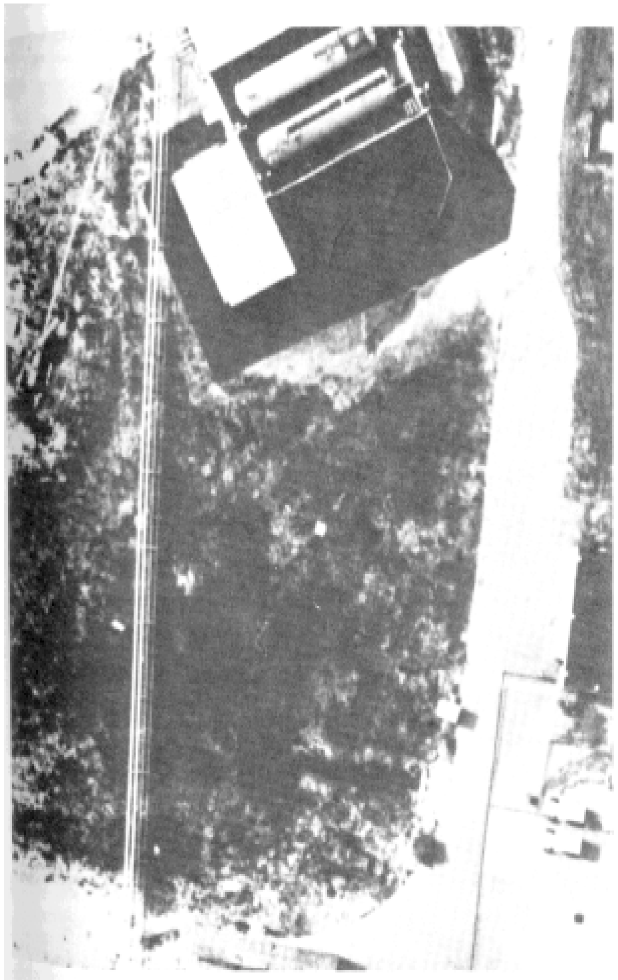


Sampling filter

sampled image



Aliasing Problems (real images!)



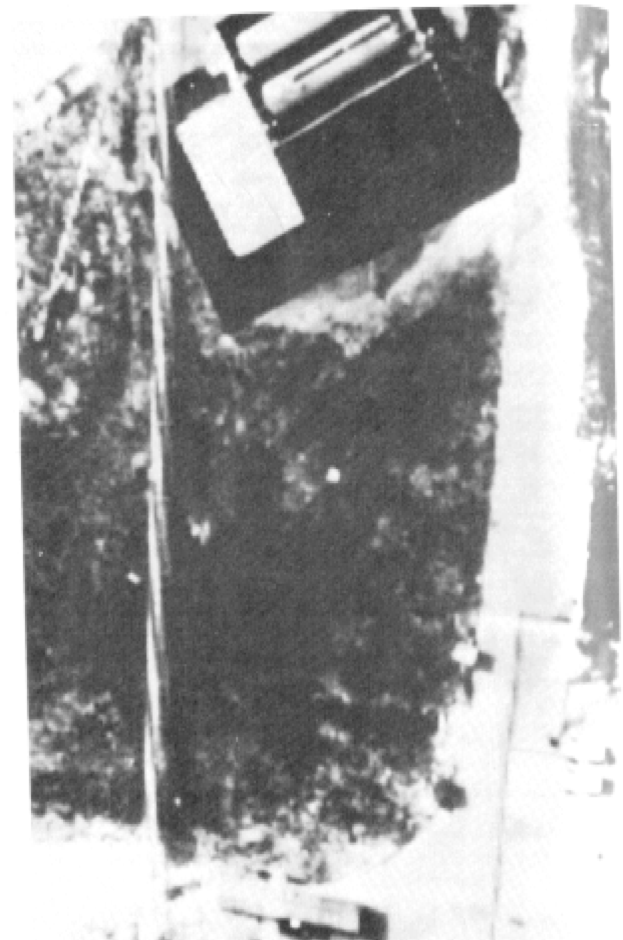


Fig. 10b Picture of Fig. 10a after sampling and reconstruction. (Courtesy Perkin-Elmer