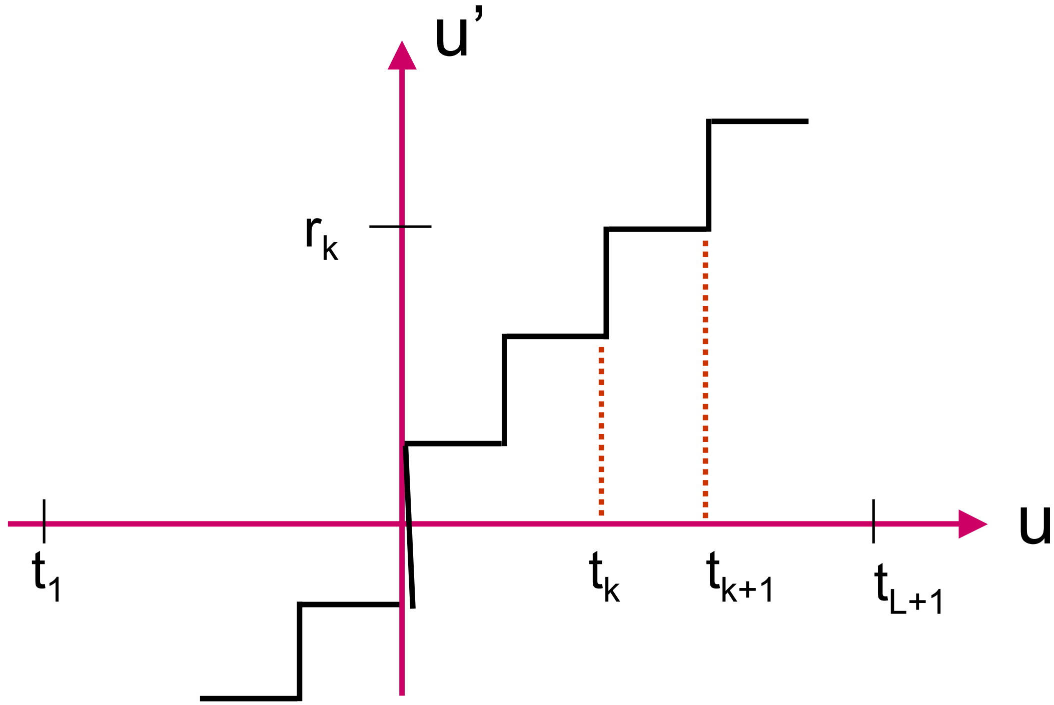
Quantization
(Jan 21, 2003)

- Optimal Quantizer
- Uniform Quantizer

Image Quantization

u (continuous) $\xrightarrow{\text{Quantizer}}$ $u' \in \{r_1, r_2, r_3, \ldots, r_L\}$



Decision/Reconstruction Levels

$u \, \varepsilon \, [t_k, t_{k+1}] \longrightarrow r_k$

$\{t_k : k=1,2,....,L+1\} \longleftarrow$ Transition or decision levels

$r_k \longrightarrow$ k^{th} reconstruction level

Example: Uniform quantizer $u \; \varepsilon \; [0,10.0]$

We want $u' \; \varepsilon \; \{0,1,.....,255\}$

$t_1 = 0$; $t_{257} = 10.0$; uniformly spaced, $t_k = (k-1).10/256$

$k = 1,2,.....,257)$

Example: quantization

$$r_k = t_k + \frac{1}{2}\left(\frac{10}{256}\right) = t_k + \frac{5}{256}$$

Quantization interval

$$q_k = t_k - t_{k-1} = r_k - r_{k-1}$$

$$= \text{Constant} \Rightarrow \text{ Uniform quantizer}$$

MMSE Quantizer

Minimise the mean squared error, MSE = Expected value of $(u-u')^2$ given the number of quantization levels L.

Assume that the density function $\mathbf{p_u(u)}$ is known (or can be approximated by a normalised histogram).

Note that for images, u==image intensity. $\mathbf{p_u(u)}$ is the image intensity ditribution.

Optimum MSE quantizer

$$\mathrm{E}(u) = \text{ Expected value of u } = \int_{-\infty}^{\infty} u \; p_u(u)\, du$$

$$MSE, \varepsilon = \mathrm{E}((u-u')^2) = \int_{-\infty}^{\infty} (u-u')^2 \, p_u(u)\, du = \int_{\mathrm{t}_1}^{t_{L+1}} (u-u')^2 \, p_u(u)\, du$$

Since $u' = r_k$ if $u \in [t_k, t_{k+1}]$, we can rewrite this as

$$\varepsilon = \sum_{i=1}^{L} \int_{t_i}^{t_{i+1}} (u-r_i)^2 p_u(u)\, du$$

Conditions for minimisation of ε are: $\dfrac{\partial \varepsilon}{\partial r_k} = 0\,; \; \dfrac{\partial \varepsilon}{\partial t_k} = 0$

MMSE (contd.)

$$\frac{\partial \varepsilon}{\partial t_k} = (t_k - r_{k-1})^2 \; p_u(t_k) \; - \; (t_k - r_k)^2 \; p_u(t_k) \quad = 0$$

$$\text{Now} \;\; t_k \le r_k < t_{k+1} \Rightarrow t_k - r_{k-1} = r_k - t_k \Rightarrow \left[t_k = \frac{(r_k + r_{k-1})}{2} \right] \text{---(A)}$$

$$\frac{\partial \varepsilon}{\partial r_k} = \int_{t_k}^{t_{k+1}} 2\,(u - r_k)\,(-1)\; p_u(u)\, du = 0$$

$$\Rightarrow r_k = \frac{\int_{t_k}^{t_{k+1}} u \; p_u(u)\, du}{\int_{t_k}^{t_{k+1}} p_u(u)\, du} \quad = \quad \mathrm{E}(u \,|\, u \in [t_k, t_{k+1}]) \qquad \text{-------(B)}$$

Optimum transition/reconst.

(1) Optimal transition levels lie halfway between the optimum reconstruction levels.

(2) Optimum reconstruction levels lie at the center of mass of the probabality density in between the transition levels.

(3) A and B are simultaneous non-linear equations (in general)

$\longrightarrow$ Closed form solutions normally don’t exist $\longrightarrow$ use numerical techniques

Uniform optimal quantizer

Consider $p_u(u) = \begin{cases} \dfrac{1}{t_{L+1} - t_1} & t_1 \le u \le t_{L+1} \\ 0 & \text{Otherwise} \end{cases}$

Then $$r_k = \frac{\int_{t_k}^{t_{k+1}} u\, p_u(u)\, du}{\int_{t_k}^{t_{k+1}} p_u(u)\, du} = \frac{\dfrac{1}{t_{L+1} - t_1} \int_{t_k}^{t_{k+1}} u\, du}{\dfrac{1}{t_{L+1} - t_1} \int_{t_k}^{t_{k+1}} du} = \frac{\left(\dfrac{u^2}{2}\right)_{t_k}^{t_{k+1}}}{(u)_t^{t_{k+1}}} = \frac{(t_{k+1}^2 - t_k^2)}{2(t_{k+1} - t_k)}$$

$$r_k = \frac{1}{2}\left(t_{k+1} + t_k\right); \quad t_k = \frac{r_k + r_{k-1}}{2} = \frac{1}{2} \cdot \frac{1}{2}\left[t_{k+1} + t_k + t_k + t_{k-1}\right] = \frac{t_{k+1} + t_{k-1}}{2}$$

Uniform Quantizer

$$t_k - t_{k-1} = t_{k+1} - t_k = \text{ Constant} \qquad q = \frac{t_{L+1} - t_1}{L}$$

$$t_k = t_{k-1} + q \quad ; \quad r_k = t_k + q/2$$

Quantization error $e = (u - u')$ is uniformly distributed over the interval

$$\left(-q/2 , q/2\right)$$

Mean squared error $E\left((u - u')^2\right) = \dfrac{1}{q}\displaystyle\int_{-q/2}^{q/2} u^2 = q^2/12$