HW #3 Solutions

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1. Test if the following systems are (a) Linear and (b) Shift-Invariant.

$(i) g[m] = \sum_{k=0}^{m} x[k]$, $g[m] = T[x[m]] = \sum_{k=0}^{m} x[k]$

a) linear if $T[ax_1[m] + bx_2[m]] = ag_1[m] + bg_2[m]$, change variables: $x'[m] = ax_1[m] + bx_2[m]$

$$T[ax_1[m]+bx_2[m]] = T[x'[m]] = \sum_{k=0}^{m} x'[k] = \sum_{k=0}^{m} ax_1[k]+bx_2[k] = a\sum_{k=0}^{m} x_1[k] + b\sum_{k=0}^{m} x_2[k] = ag_1[m]+bg_2[m]$$

system is linear

b) shift-invariant if $T[x[m-m_0]] = g[m-m_0]$.

Change variables: $x'[m] = x[m-m_0]$

$$T[x[m-m_0]] = T[x'[m]] = \sum_{k=0}^{m} x'[k] = \sum_{k=0}^{m} x'[k-m_0] = \sum_{k'=-m_0}^{m-m_0} x'[k'] \neq \sum_{k'=0}^{m-m_0} x'[k'] = g[m-m_0]$$

system is not shift invariant

NOTE: for x[m]=0, m<0 system is shift-ivariant but that is not the general case

$(ii) g[m,n] = (m+n)x[n]$, $g[m,n] = T[x[m,n]] = (m+n)x[m,n]$

a) linear if $T[ax_1[m]+bx_2[m]] = ag_1[m]+bg_2[m]$.

Change variables: $x'[m] = ax_1[m]+bx_2[m]$

$$T[ax_1[m,n]+bx_2[m,n]] = T[x'[m,n]] = (m+n)x'[m,n] = (m+n)(ax_1[m,n]+bx_2[m,n]) =$$

$= a(m+n)x_1[m,n]+b(m+n)x_2[m,n] = ag_1[m]+bg_2[m]$, **system is linear.**

b) shift-invariant if $T[x[m-m_0, n-n_0]] = g[m-m_0, n-n_0]$, change variables:

$x'[m,n] = x[m-m_0, n-n_0]$

$$T[x[m-m_0, n-n_0]] = T[x'[m,n]] = (m+n)x'[m,n] = (m+n)x[m-m_0, n-n_0] \neq$$

$\neq (m-m_0+n-n_0)x[m-m_0, n-n_0] = g[m-m_0, n-n_0]$, **system is not shift invariant**

3. The image $f(x, y) = 4\cos 2\pi(2x + y)$ is to be sampled such that one can reconstruct the signal from its samples without any errors. Suggest a sampling scheme. How do you propose to reconstruct from the samples? Is your sampling scheme optimal (in the sense that it requires fewest number of samples to reconstruct the signal error free)?

$$f(x, y) = 4\cos 2\pi(2x + y) \Rightarrow u_0 = 2, v_0 = 1$$

$$F(u,v) = 2(\delta(u+2, v+1) + \delta(u-2, v-1))$$

$$u_s = \frac{1}{\Delta x}, u_s > 2u_0 \Rightarrow u_s > 4 \Rightarrow \Delta x = 0.25$$

$$v_s = \frac{1}{\Delta y}, v_s > 2v_0 \Rightarrow v_s > 2 \Rightarrow \Delta y < 0.5$$

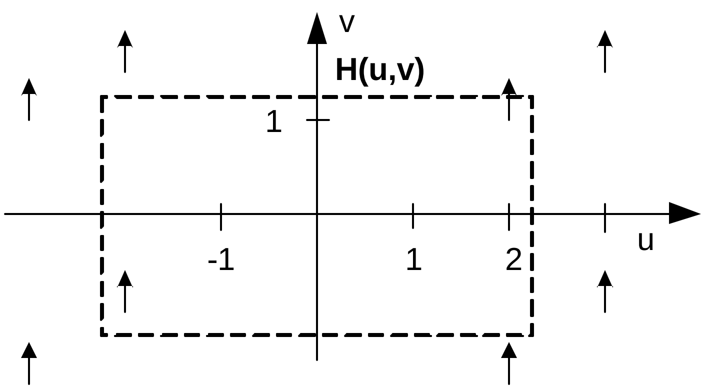
To avoid aliasing, pick $\Delta x, \Delta y$ smaller than Nyquist sampling rate. Lets pick $\Delta x = 0.2, \Delta y = 0.4$, which corresponds to $u_s = 5, v_s = 2.5$. Then:

$$F_s(u,v) = 250 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (\delta(u+2-5k, v+1-2.5l) + \delta(u-2-5k, v-1-2.5l))$$

NOTE: In the optimal sampling scheme, and using LPF for reconstruction, $\Delta x, \Delta y$ are determined by the selectivity of LPF.

Lets pickl the reconstructed filter *H(u,v)* as the ideal low pass filter corresponding to the above sampling rate, i.e.

$$H_s(u,v) = \begin{cases} 0.08, & \begin{matrix} -2.5 \le u \le 2.5 \\ -1.25 \le v \le 1.25 \end{matrix} \\ 0, & \text{otherwise} \end{cases}$$

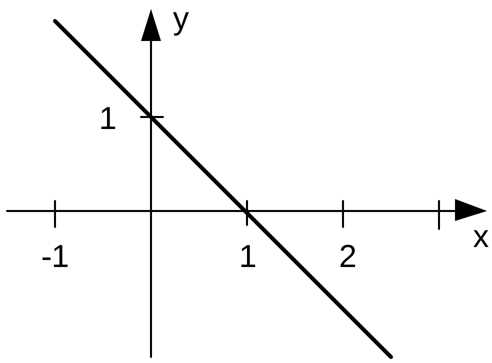


Reconstructed signal is:

$$\hat{F}(u,v) = H(u,v) F_s(u,v) = (125 * 0.08) 2(\delta(u+2, v+1) + \delta(u-2, v-1))$$

Perfect reconstruction!

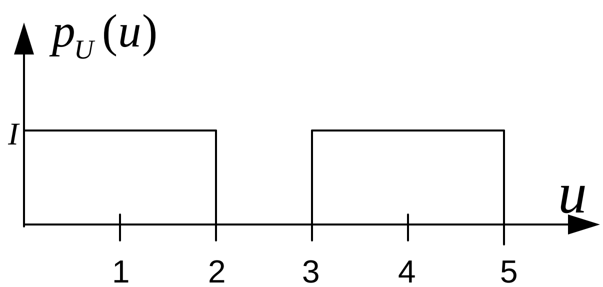
4. Sketch the impulse function $\phi(x,y)=\delta(x+y-1)$



$\delta(t)=1 \Leftrightarrow t=0$

$t=x+y=1,\Rightarrow x+y-1=0\Rightarrow y=1-x$

5. Design a four level minimum mean squared error quantizer for u where the signal has the following pdf. What is the quantization error?



-MMSE quantizer:

$$r_k=\frac{\int_{t_k}^{t_{k+1}} u p_U(u)du}{\int_{t_k}^{t_{k+1}} p_U(u)du},\ t_k=\frac{r_{k-1}+r_k}{2}$$

$$-\int_u p_U(u)du=1\Rightarrow I=\frac{1}{4}$$

-pdf symmetry: $\boxed{t_1=0}$, $\boxed{t_3=2.5}$, $\boxed{t_5=5}$, $t_5-t_4=t_2-t_1\Rightarrow t_4=5-t_2$ (1)

$$-\ r_1=\frac{\int_0^{t_2} u\frac{1}{4}du}{0\int_{t_k}^{t2}\frac{1}{4}du}=\frac{t_2}{2}\Rightarrow r_1=\frac{t_2}{2}, t_2=\frac{r_1+r_2}{2}\ (2) \qquad -r_2=\frac{\int_{t_2}^{t_3} u p_U(u)du}{\int_{t_2}^{t_3} p_U(u)du}=\frac{\int_{t_2}^{2} u\frac{1}{4}du}{\int_{t_2}^{t_2}\frac{1}{4}du}=\frac{2+t_2}{2}\ (3)$$

$$-\ (2)\ \&\ (3)\ \Rightarrow t_2=\frac{r_1+r_2}{2}\Rightarrow t_2=\frac{1}{2}\left(\frac{t_2}{2}+\frac{2+t_2}{2}\right)\Rightarrow \boxed{t_2=1} \qquad -\ (1)\Rightarrow t_4=t_5-t_2\Rightarrow \boxed{t_4=4}$$

$$-\ (2)\ \ r_1=\frac{t_2}{2}\Rightarrow \boxed{r_1=\frac{1}{2}} \qquad -\ (3)\ \ r_2=\frac{2+t_2}{2}\Rightarrow \boxed{r_2=\frac{3}{2}}$$

$$-\ \text{pdf symmetry:}\ \ r_3=5-r_2\Rightarrow \boxed{r_3=\frac{7}{2}},\ r_4=5-r_1\Rightarrow \boxed{r_3=\frac{9}{2}}$$

- This 4-level MMSE quantizer combines two uniform 2-level quantizers ([0,2]&[3,5]), and $\Delta=1$

$$\Rightarrow \varepsilon=\frac{\Delta^2}{12}\Rightarrow \boxed{\varepsilon=\frac{1}{12}}$$