First-order circuits - review

\[ V(t) + RC \frac{dV}{dt} = V(t) \]

Let us restrict ourselves now to problems with zero initial conditions

Laplace circuit

\[ V_{in}(s) + \frac{1}{sRC} V_{out}(s) = V_{out}(s) \]

\[ V_{out}(s) = V_{in}(s) \frac{1}{1 + s \tau} \]

where \( \tau = RC \)

\[ H(s) = \text{transfer function} = \frac{V_{out}(s)}{V_{in}(s)} \]
Sinusoidal response

We can work with complex exponentials, e.g. \( \exp(j\omega t) \) as stimulus and (implicitly) take the real part of both stimulus and response.

**Input voltage** = \( v_{in}(t) = v_{in} \cdot e^{j\omega t} \)

**Output voltage** = \( v_{out}(t) = V_{out} \cdot e^{j\omega t} \)

where \( v_{in} \) & \( V_{out} \) are complex numbers.

\[ \Rightarrow v_{out} = H(j\omega) \cdot v_{in} \]

In our example: \( V_{out} = \frac{v_{in}}{1 + j\omega} \)

Specifically, the amplitude of the output is \( |H(j\omega)| \) times the input amplitude.

And

The phase of the output leads the phase angle of the input by an angle

\[ \Rightarrow \theta = |H(j\omega)| \]
**Bode plots**

- Represent the amplitude and phase of $H(j\omega) \rightarrow H(j2\pi f)$ vs. frequency.
- Vertical axes: $\mathrm{dB} = 10 \log_{10}$ (power ratio) $\rightarrow 20 \log_{10}$ (voltage ratio).

Example: $H(j\omega) = (1 + j\omega T)^{-1}$; $T = 159.125$

$H(j2\pi f) = (1 + jf/f_{\text{pole}})^{-1}$ where $f_{\text{pole}} = 1 \mathrm{GHz}$

= "Asymptotic" (straight-line) plot is often more informative than actual curve.

= Please use semi-log paper correctly!
More complex Bode amplitude plots are very clear in asymptotic fam:

$\| H(j \omega) \|$ dB

\begin{figure}
\centering
\includegraphics[width=\textwidth]{bode_plot}
\caption{Bode Plot Example}
\end{figure}

Example:

$H(j \omega) = \frac{(1 + j \omega / f_{p1})}{(1 + j \omega / f_{p1})(1 + j \omega / f_{p2})}$

$f_{p1} = 100$ kHz

$f_{p2} = 3.6$ kHz

$f_{f1} = 1.6$ kHz
Bode phase plot

This is $|H(j\omega)|$ plotted vs frequency, using a
semi-log axis for frequency.

\[ \frac{1}{|H(j\omega)|} \]

\[ -90 \]

\[ -45 \]

\[ 0 \]

Plot for

\[ f_p = 1 \text{ GHz} \]

For a single-pole transfer function:

\[ H(j\omega) = \frac{1}{1 + j\omega f_p} \]

\[ \frac{1}{H(j\omega)} = -\arctan(\omega f_p) = -\arctan(f/f_p) \]

The exact plot is well-approximated by 3 asymptotes
as indicated.
Root locus as a graphical tool to predict magnitude-phase characteristics:

\[ H(\omega) = \frac{1}{1 + j\omega} = \frac{1}{\omega + \text{Ap}} \], where \( \text{Ap} = -\text{Ap} \)

\[ \text{A-plane} \]
\[ \text{A} = r + j\omega \]

\[ \| H(\omega) \| = \frac{1}{\| \omega - \text{Ap} \|} \]

\[ \Rightarrow \text{Note that } \| H(\omega) \| \text{ varies as the inverse of the length of the vector between A and Ap:} \]

\[ \text{vector} = A - \text{Ap} \]
\[ A = j\omega \]
we can therefore plot $H(j\omega)$ from the Root locus.

From the construct:

... it is clear that the transfer function must be reduced $1/\sqrt{2}$ (e.g., $-3 dB$) when $f = Ap$.
The root locus similarly allows phase to be found:

\[ H(j\omega) = \frac{1}{1 + j\omega} = \frac{1}{\frac{1}{\omega} - \omega} \]

\[ H(j\omega) = -\frac{1}{\omega - A_p} \]

Once again - it is clear that \( \angle H = -45^\circ \) at the frequency equal to the pole frequency.

Clearly, the geometry shows that phase must vary from zero to \(-90^\circ\) as \( \omega \) varies from zero to infinity.
root locus for a more complex transfer function:

\[ H(j\omega) = \frac{A \cdot 2a}{(\alpha - j\omega_0 + \alpha)(\alpha + j\omega_0 + \alpha)} \]

Can you see why the magnitude must show this strong response near \( \omega = \omega_0 \)?
Now consider the transient response:

\[ V_{in}(t) = R \frac{dV(t)}{dt} + V(t) \]

\[ V_{out}(s) = H(s) \frac{V_{in}(s)}{V_{out}(s)} = \frac{1}{1 + s \tau} \]

Impulse response:

\[ V_{in}(t) = k \cdot \delta(t) \]

Note: \( \delta(t) \) has units of \( 1/\text{time} \)

\( k \) has units of volts \( \times \) time

\[ V_{in}(s) = k \]

Consistent units, as \( V(s) \rightarrow \text{volts} \times \text{time} \).

\[ V_{out}(s) = \frac{k}{1 + s \tau} \]

\[ V_{out}(t) = k e^{-\frac{t}{\tau}} \cdot u(t) \rightarrow \text{note: units again check.} \]

Input

Output:

Duration @ 50% point = \( \tau \ln 2 \)
Step response

\[ v(t) = v_0 \cdot u(t) \quad \text{units: \textit{volts}} \]

\[ v_{in}(t) = v_0 t \quad \text{units: \textit{volts \times time}} \]

\[ v_{out}(t) = \frac{v_0}{\tau} \left( 1 - \frac{v_0 t}{\tau} \right) \]

\[ v_{out}(t) = v_0 \cdot u(t) - v_0 \cdot u(t) \cdot e^{-t/\tau} \]

\[ = v_0 \cdot (1 - e^{-t/\tau}) \cdot u(t) \]

10% - 90% rise time: \[ T_{10-90} = \frac{\tau}{\ln(2)} \left[ u(0.9) - u(0.1) \right] \]

\[ = 2.2 \cdot \tau \]
So for a single-pole system

\[ \| H(j\omega) \| = \sqrt{\frac{1}{1 + \frac{\omega^2}{\omega_p^2}}} \quad \omega_p = \frac{1}{T} \]

-3 dB when \( \omega = \frac{1}{T} \) (\( f = f_p \))

this is the 3-dB bandwidth \( f_{3dB} \)

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Step response risetime = 2.2 \( \tau \)

\[ T_{10\% - 90\%} = 2.2 \tau \]

\[ = 2.2 \left( \frac{2}{f_{3dB}} \right) \]

\[ = 2.2 \left( \frac{f_{3dB}}{2\pi} \right) \]

\[ T_{10 - 90} = 0.35 \left( \frac{1}{f_{3dB}} \right) \]

true only for a single pole system.