ECE 137B Notes Set 6: Method of time constants

General circuit transfer function:

\[ \frac{V_o}{V_{in}} = \frac{1}{1 + a_1 V_{in} + a_2 V_{in}^2 + \ldots} \]

Methods of Solution

- Nodal analysis: hard to do for complex circuits, answers often hard to interpret
- Miller approximations: intuitive (often very wrong)
- SPICE: equivalent to randomly building and testing, zero understanding.
- SPICE is for design verification, not synthesis.

- Time constant method.
  - exact - for the information it gives easy and fast
  - answers often physically interpretable
Let us start with a specific circuit:

Turn off independent voltage sources → short circuits
Turn off independent current sources → open circuits
Dependent sources must be kept.
General picture: $C_2$ ports.

Define $R_{ii}^0$: resistance at port 1 with all capacitors replaced with open circuits.

$R_{ii}^0 = \frac{V}{I}$

(one can also force a voltage & measure $I$)

*inductors, transmission lines, time delay elements, or anything with frequency dependence is not allowed.
It can be shown that

$$a_i = R_{ii}C_i + R_{22}C_2 + R_{33}C_3 + \ldots$$

These are called the open circuit time constants.
What is $R_{11}^o$? - Easy!

$R_{11}^o \Rightarrow R_i$  $\Rightarrow R_{11}^o = R_i$

So, first time constant is $CgsR_i$.

What is $R_{33}^o$? - Easy!

$R_{33}^o = R_{leg}$

So, third time constant is $R_{leg}C_2$. 
what is $R_{22}$? - harder

we are faced with analyzing this problem:

\[ R_i \quad \text{+} \quad V_i \quad \text{=} \quad R_{\text{reg}} \]

- this situation is going to show up with common emitter/source and common collector/drain.

- Let us analyze the problem once in a general way.

\[ R_i = \frac{V_{\text{in}}}{R_{\text{in}}} \]

\[ j_{\text{in}} \cdot V_{\text{in}} \]

\[ R_{\text{reg}} = R_{\text{out}} \]

or:

\[ R_{\text{in}} \quad \text{+} \quad V_{\text{in}} \quad \text{=} \quad R_{\text{out}} \quad \text{+} \quad \frac{V_{\text{in}}}{R_{\text{out}}} \]

\[ A_V \cdot V_{\text{in}} \quad \text{=} \quad -j_{\text{m}} \cdot R_{\text{reg}} \]
or:

- $V_{in} = R_{in} \cdot I$
- Voltage at amplifier output = $+A_{v}R_{in} \cdot I$
- Voltage drop across $R_{out} = I \cdot R_{out}$

$\Rightarrow V = I \cdot R_{in} - I \cdot R_{in} \cdot A_{v} + I \cdot R_{out}$

$\Rightarrow R_{22}^\circ = R_{in} (1 - A_{v}) + R_{out}$ general answer, similar to Miller effect

For our specific common-source example:

$R_{22}^\circ = R_{i} (1 + g_{m} R_{eg}) + R_{eg}$

We will use the general form several times.
So the second time constant is $C_{gd} \cdot R_{set}$

e.g. $C_{gd} \left[ \frac{1}{R_1 (1+g_m \cdot R_{reg})} + R_{reg} \right]$

The sum of time constants is

$a_1 = R_1 \cdot C_{gy} + R_{reg} \cdot C_2 + C_{gd} \left[ \frac{1}{R_1 (1+g_m \cdot R_{reg})} + R_{reg} \right]$

Recognize that this is the same as that found by nodal analysis.
What about higher order poles?

\[ \frac{v_0}{v_{gen}} = \frac{v_0}{v_{gen}} \left( \frac{1 + b_1 A + b_2 A^2 + \ldots}{1 + a_1 A + a_2 A^2 + \ldots} \right) \]

1) If \( a_3 \) and \( a_4 \) etc are small

- and -

2) If \( a_2/A_1 \ll a_1 \)

- then -

\[ \frac{v_0}{v_{gen}} = \frac{v_0}{v_{gen}} \left( \frac{1 + b_1 A + b_2 A^2 + \ldots}{v_{gen}/m_b (1 + a_1 A) (1 + a_2 A)} \right) \]

(Separated pole approximation)

Whether or not we use the SPA,

- how do we find \( a_2 \)?
\[ A_2 = R_{ii} C_l C_2 R_{22} + R_{ii} C_l C_3 R_{33} + R_{22} C_2 C_3 R_{33}^2 \]

(can be applied this way to more than \( n \))

Where we define \( R_{22}' \) as below:

\[ R_{22}' = "\text{Resistance at port } 2\text{ with port 1 short-circuited but all other ports open-circuited}" \]
Now consider a 2-port for a moment:

\[ \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \]

\[ R_{11}^0 = Z_{11} = \frac{V_1}{I_1} \bigg|_{I_2=0} \quad \text{and} \quad R_{11}^2 = \frac{V_1}{I_1} \bigg|_{V_2=0} \]

\[ \text{If } V_2 = 0 \Rightarrow Z_{21} I_1 + Z_{22} I_2 = 0 \Rightarrow V_1 = \frac{Z_{11} I_1}{I_1} = \frac{Z_{11}}{Z_{22}} = R_{11}^2 \]

Similarly: \[ R_{22}^2 = \frac{Z_{22} - Z_{12} Z_{21}}{Z_{11}} \]

So: \[ R_{11}^0 R_{22}^1 = Z_{11} \left[ \frac{Z_{22} - Z_{12} Z_{21}}{Z_{11}} \right] = Z_{11} Z_{22} - Z_{12} Z_{21} = \Delta Z \]

and \[ R_{11}^2 R_{22}^0 = \left[ \frac{Z_{11} - Z_{22} Z_{21}}{Z_{22}} \right] Z_{22} = \ldots = \Delta Z \]

So: \[ R_{11}^0 R_{22}^1 = R_{22}^0 R_{11}^2 = \Delta Z \]
So:

\[ a_2 = R_1^0 C_1 C_2 R_2 + R_1^0 C_1 C_3 R_3 + R_2^0 C_2 C_2 R_3^2 \]

\[ = R_1^2 C_1 C_2 R_2^0 + R_1 R_1 C_2 R_3^0 + R_2^0 C_2 C_3 R_3^0 \]

Each term can be calculated two ways.

Returning to our example:

\[ \begin{align*}
C_1 R_1^0 R_2 R_1^2 C_2 & \quad R_1^0 = R_1 \\
C_1 R_1^0 R_3 R_1^2 C_2 & \quad R_2^0 = R_2 \\
C_2 R_2^0 R_3 R_2^2 C_3 & \quad R_3^0 = R_3 \\
\end{align*} \]

Draw sketches for each case:

\[ R_2^0 = R_2 \]

\[ R_3^0 = R_3 \]

\[ C_2 R_2^0 R_3^0 C_3 = C_2 R_2^2 R_3^0 C_3 \]

\[ R_3^0 = R_3 \quad R_2^0 = R_2 \]
So: \( R_{11}' R_{22}' = R_{11}' R_{33}' = R_{21}' R_{33}^2 \) for this case only

\[ = R \cdot \text{Rlog} \]

hence: \( a_1 = R \cdot \text{Rlog} \left[ C_{gs} C_{gd} + C_{gs} C_L + C_{gd} C_L \right] \)

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For our example

\[ V_{OL}/V_{GEN} = V_{OL}/V_{GEN} \left\{ \frac{1 + \text{mid}}{1 + A_1 A + A_2 A^2} \right\} \]

\[ a_1 = R \cdot C_{gs} + R \cdot \text{Rlog} C_L + \left[ R \cdot (1 + g_m \text{Rlog}) + \text{Rlog} \right] C_{gd} \]

\[ a_2 = R \cdot \text{Rlog} \left[ C_{gs} C_{gd} + C_{gs} C_L + C_{gd} C_L \right] \]

... as was found earlier by nodal analysis.