ECE 145A / 218 C, notes set 13: Very Short Summary of Noise

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Background / Intent

Noise is a complex subject

- Math
- Physics
- Device noise models
- Circuit noise analysis
- 2 - port noise representations
- Systems noise analysis
- System sensitivity calculations

We can only touch upon a very few points here.

More details in other classes.

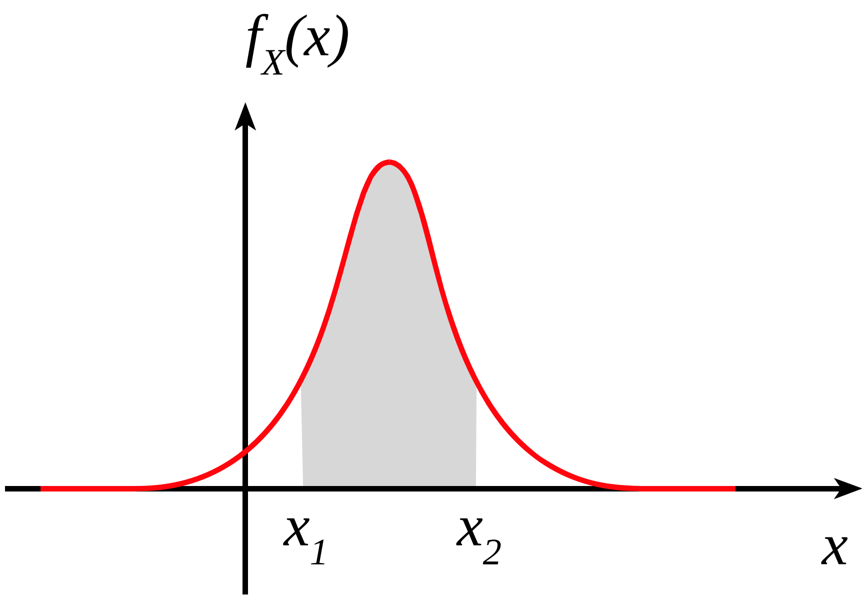
Probability Distribution Function: 1 Random Variable

During an experiment, a random variable X takes on a particular value x.

The probability that x lies between x_1 and x_2 is

$$P\{x_1 < x < x_2\} = \int_{x_1}^{x_2} f_X(x)dx$$

$f_X(x)$ is the probability distribution function.



Mean value, Variance, Standard Deviation

$$\text{Mean Value of X}: \ \langle X \rangle = \overline{X} = E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

$$\text{Expected value of } X^2: \quad \langle X^2 \rangle = E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx$$

The variance σ_x^2 of X is its root-mean-square deviation from its average value

$$\sigma_X^2 = \left\langle (X - \overline{x})^2 \right\rangle = E\left[(X - \overline{x})^2\right] = \int_{-\infty}^{+\infty} (x - \overline{x})^2 f_X(x)\,dx$$

$$\sigma_x^2 = \langle X^2 \rangle - (\overline{x})^2.$$

The standard deviation σ_x of X is simply the square root of the variance

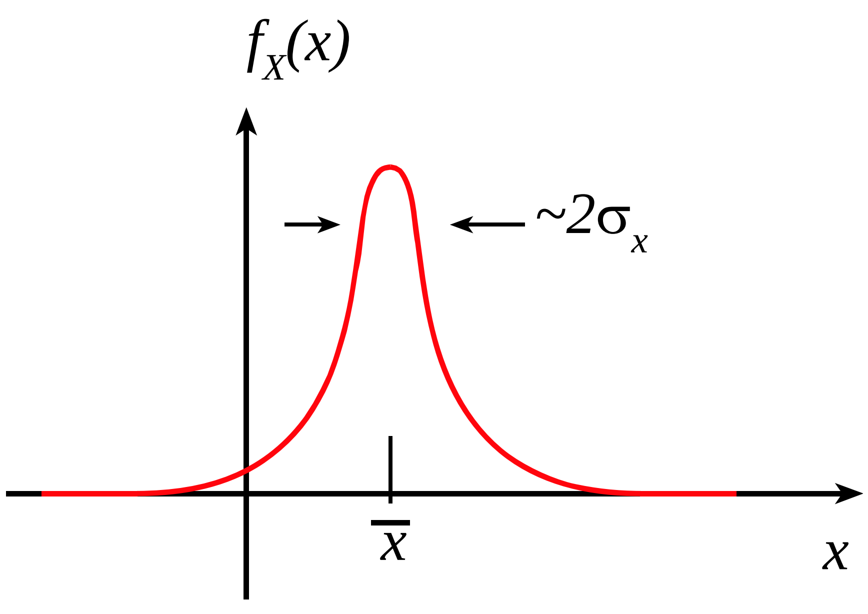
The Gaussian Distribution

The Gaussian distribution :

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(\frac{-(x-\overline{x})^2}{2\sigma_x^2} \right)$$

The mean $(\overline{x})$ and the standard deviation (σ_x^2) are as defined previously.

Because of the *central limit theorem*, physical random processes arising from the sum of many small effects have probability distributions close to that of the Gaussian.



Pair of Random Variables

In an experiment, a pair of random variables X and Y takes on specific particular values x and y.

Their joint behavior is described by the joint distribution $f_{XY}(x, y)$

$$P\{A < x < B \text{ and } C < y < D\} = \int_C^D \int_A^B f_{XY}(x, y)\,dx\,dy$$

Correlation and Covariance

The correlation of X and Y is

$$R_{XY} = E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy \bullet f_{XY}(x,y)\,dxdy$$

The covariance of X and Y is

$$C_{XY} = E[(X-\bar{x})(Y-\bar{y})] = R_{XY} - \bar{x}\cdot\bar{y}$$

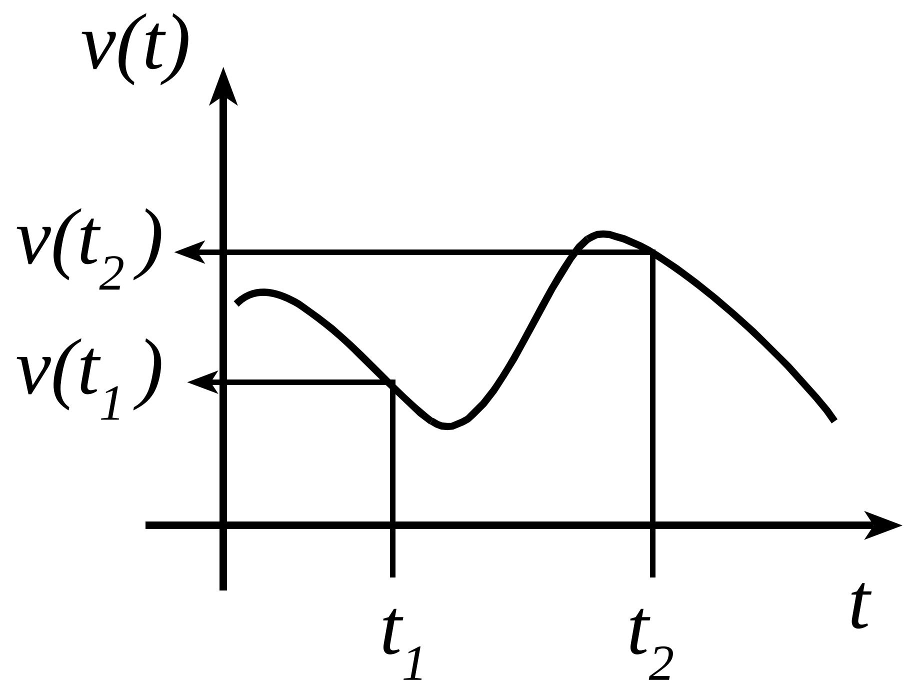
Note that correlation and covariance are the same if either X or Y have zero mean values.

If we are dealing with AC noise signals (subtracting off DC bias), the mean values are zero.

Random Processes

A voltage $V(t)$ varying (in some sense) randomly with time.

Measured at times t_1 and t_2, $V(t)$ has values $V(t_1)$ and $V(t_2)$.



Autocorrelation Function of A Random Process

Write for simplicity $V_1 = V(t_1), V_2 = V(t_2)$

$R_{VV}(t_1, t_2) = E[V_1, V_2]$

If the process is stationary, the time origin does not matter,

and $E[V(t_1), V(t_2)] = E[V(t_3), V(t_3 + t_2 - t_1)] = E[V(t), V(t + \tau)]$

Then $R_{VV}(\tau) = E[V(t), V(t + \tau)]$

Power Spectral Density of a Random Process

Power spectral density = Fourier transform of $R_{VV}(\tau)$:

$$S_{VV}(\omega) = \int_{-\infty}^{+\infty} R_{VV}(\tau)\exp(-j\omega\tau)d\tau$$

$R_{VV}(\tau)$ and $S_{VV}(\omega)$ are a transform pair, so $R_{VV}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{VV}(\omega)\exp(j\omega\tau)d\omega$

The power delivered to a load is V^2 / R_{load},

so the average value of the noise power is $E[V^2]/R_{load}$

But $E[V^2] = R_{VV}(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{VV}(\omega)d\omega$

So, integrating the power spectral density (& dividing by R_{Load})

gives us the noise power.

Single-Sided Hz-based Spectral Densities

Double - Sided Spectral Densities

$$R_{VV}(\tau) = E[V(t)V(t+\tau)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{VV}(j\omega)\exp(j\omega\tau)d\omega$$

$$S_{VV}(j\omega) = \int_{-\infty}^{+\infty} R_{VV}(\tau)\exp(-j\omega\tau)d\tau$$

Single - Sided Hz - based Spectral Densities

$$R_{VV}(\tau) = E[V(t)V(t+\tau)] = \frac{1}{2}\int_{-\infty}^{+\infty} \tilde{S}_{VV}(jf)\exp(j2\pi f\tau)df$$

$$\tilde{S}_{VV}(jf) = 2\int_{-\infty}^{+\infty} R_{VV}(\tau)\exp(-j2\pi f\tau)d\tau$$

Single-Sided Hz-based Spectral Densities- Why ?

Why this notation ?

The signal power in the bandwidth $\{f_{\text{low}}, f_{\text{high}}\}$

$$\text{Power} = \frac{1}{2R_{Load}} \int_{-f_{\text{high}}}^{-f_{\text{low}}} \tilde{S}_{VV}(jf)df + \frac{1}{2R_{Load}} \int_{f_{\text{low}}}^{f_{\text{high}}} \tilde{S}_{VV}(jf)df = \frac{1}{R_{Load}} \int_{f_{\text{low}}}^{f_{\text{high}}} \tilde{S}_{VV}(jf)df$$

$\rightarrow \tilde{S}_{VV}(jf)/R_{Load}$ is directly the Watts of signal power per Hz of signal bandwidth at frequencies near to f.

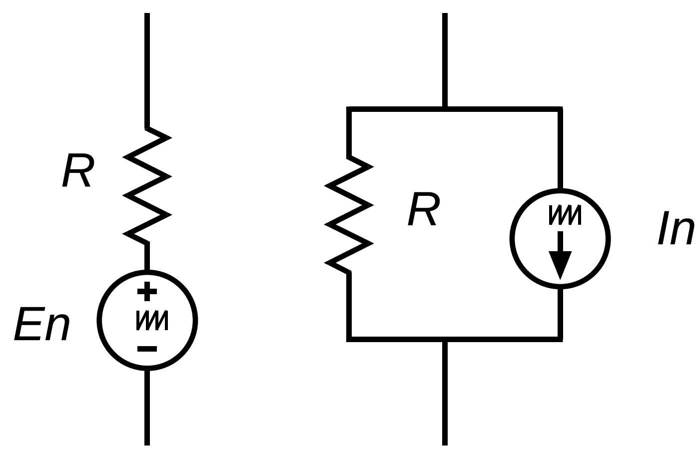
Thermal Noise: Resistances

For $hf \ll kT$, resistors produce noise voltage.

$$\tilde{S}_{VV}(jf) = 4kTR$$

By a Thevenin - Norton transformation, the noise current is

$$\tilde{S}_{II}(jf) = \frac{4kT}{R}$$

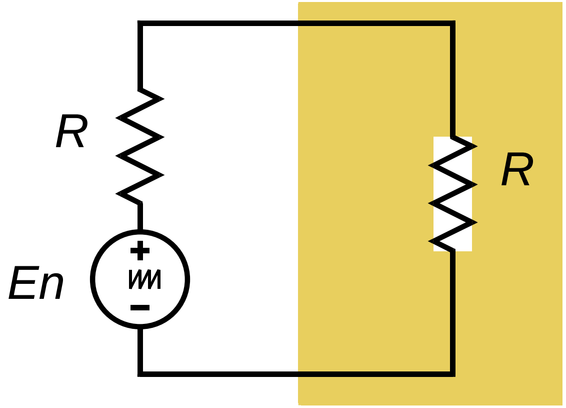


Available Thermal Noise Power

Maximum power transfer : load R matched to generator R.

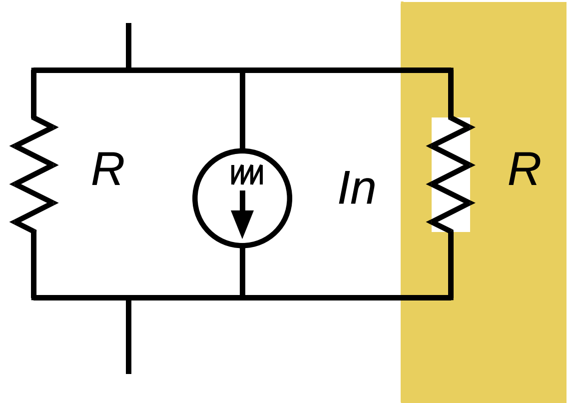
With matched load, voltage across load is $E_N / 2$

With matched load, current through load is $I_N / 2$



Given that

$$\tilde{S}_{VV}(jf) = 4kTR \text{ or } \tilde{S}_{II}(jf) = \frac{4kT}{R} \quad \Rightarrow \quad \frac{d\langle P_{load} \rangle}{df} = kT$$



P_{load} is the maximum (the available) noise power, hence in a bandwidth Δf, $P_{available,noise} = kT \cdot \Delta f$

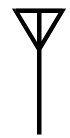
All resistors have equal available noise power.

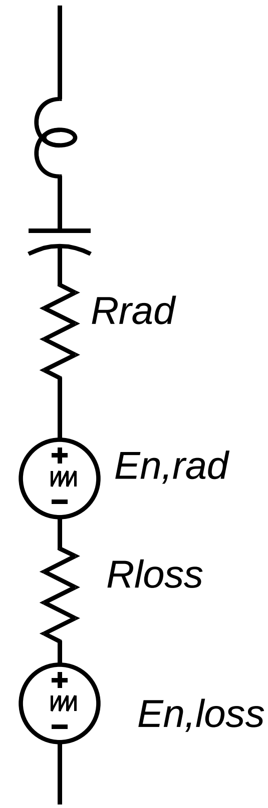
Any compononent under thermal equilibrium (no bias) follows this law.

Noise from an Antenna

$$\frac{d\langle P_{available,noise}\rangle}{df} = kT \Rightarrow \tilde{S}_{VV}(jf) = 4kT\,\mathrm{Re}(Z)$$

The antenna has both Ohmic and radiation resistances.





The Ohmic resistance has a noise voltage of spectral density $4kT_{ambient}R_{Ohmic}$, where $T_{ambient}$ is the physical antenna temperature

The radiation resistance has a noise voltage of spectral density $4kT_{field}R_{rad}$, where T_{field} is the average temperature of the region from which the antenna receives signal power

Inter-galactic space is at 2.7 Kelvin (Big Bang)

Shot noise: PN junctions

If, given a DC current I_{DC}, the arrival of each electron is statistically independent of every other electron,

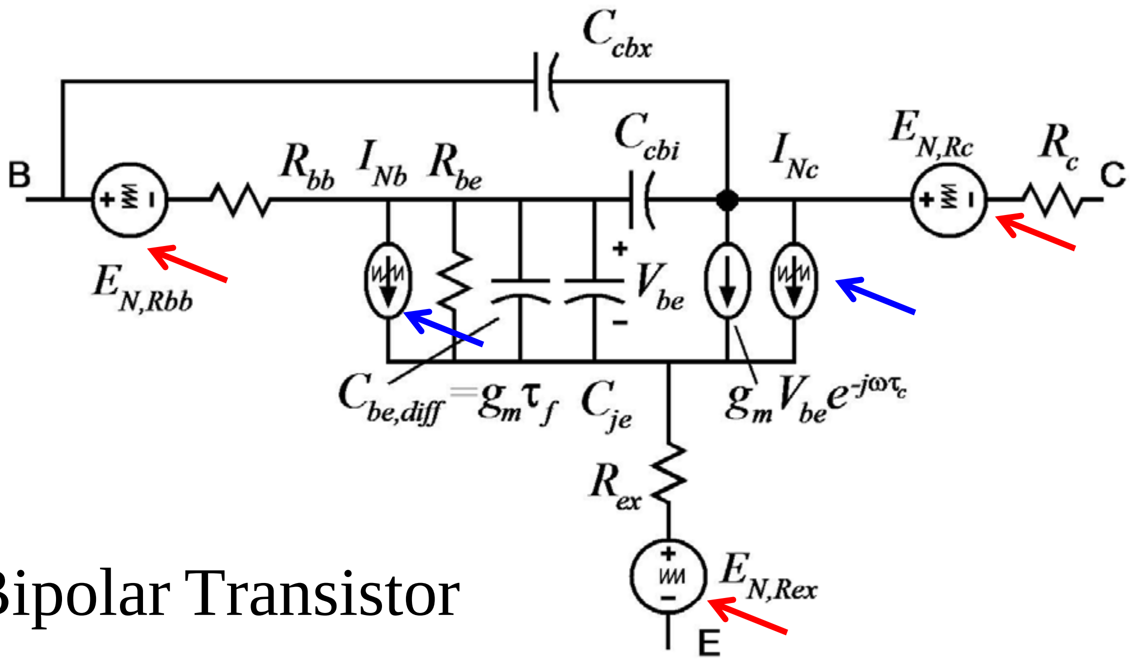
then the DC current has a noise fluctuation I of power spectral density

$$\tilde{S}_{II}(jf) = 2qI_{DC}$$

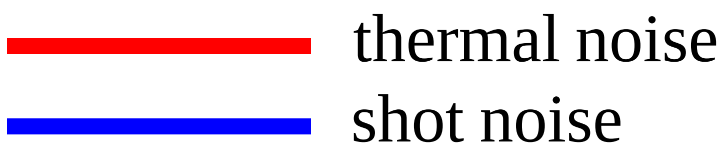
DC currents in resistors are *not* a statistically independent flow of electrons, and *do not* generate short noise.

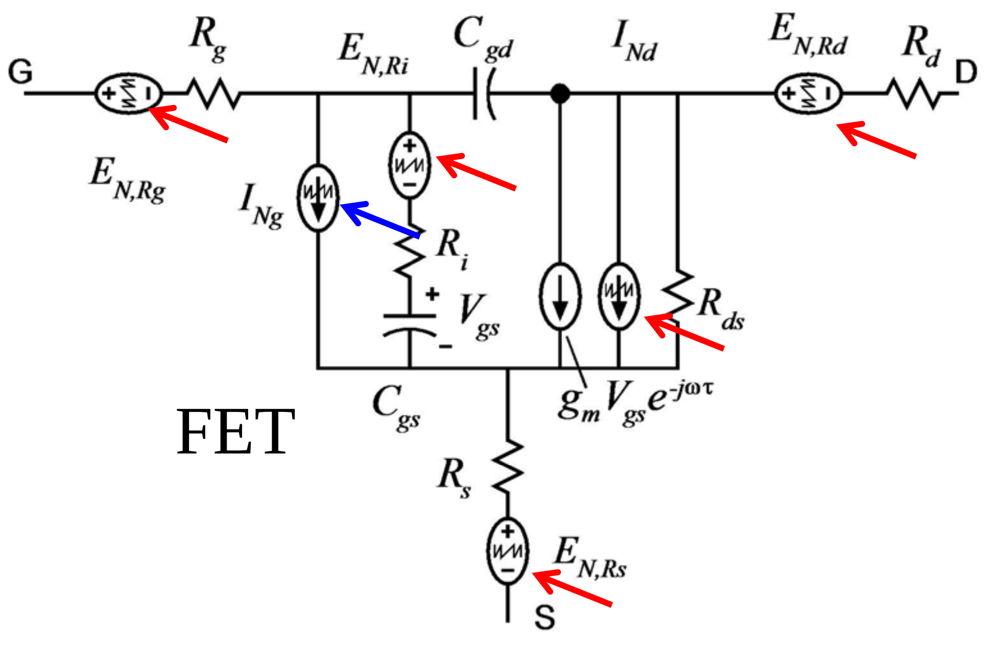
Currents in PN junctions *DO* exhibit shot noise.

Device Noise Models: Descriptive, No Explanation



Bipolar Transistor

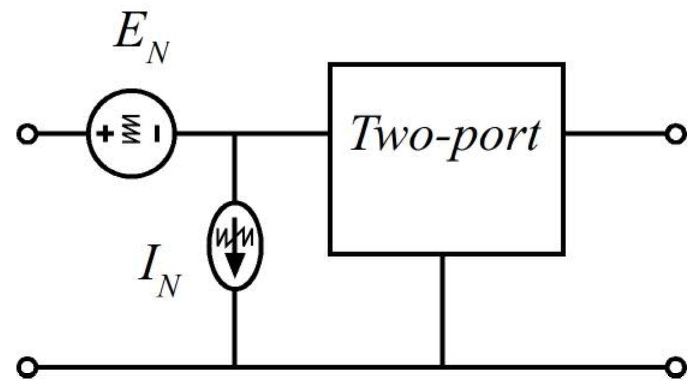




FET

Two-Port Noise Description

Through the methods of circuit analysis, the internal noise generators of a circuit can be summed and represented by two noise generators E_n and I_n.



The spectral densities of E_n and I_n must be calculated and specified. The cross spectral density * must also be calculated and specified.

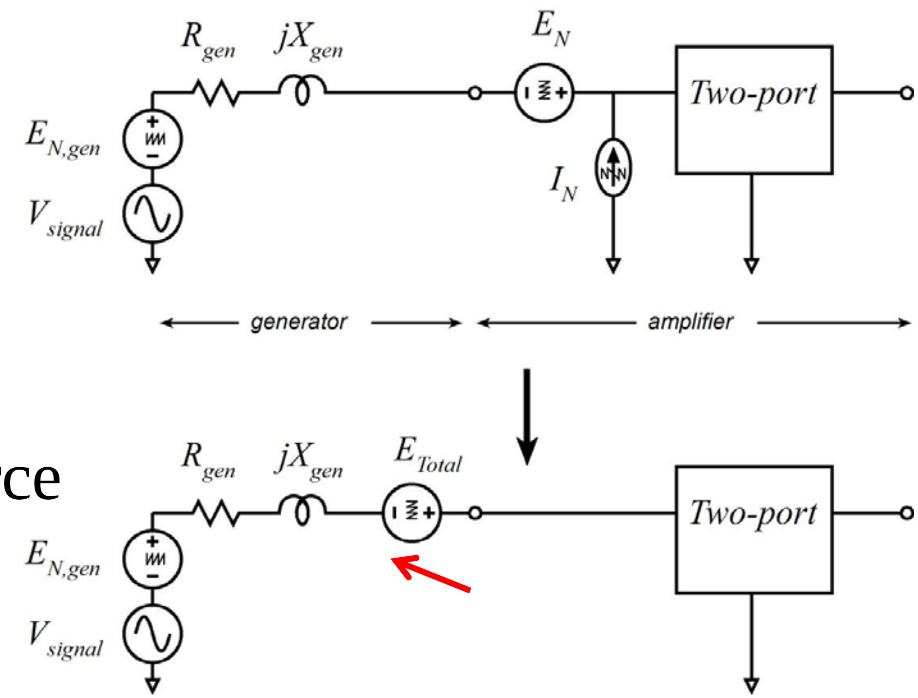
* Cross spectral density : $\tilde{S}_{VI}(jf) = 2\int_{-\infty}^{+\infty} R_{VI}(\tau)\exp(-j2\pi f\tau)d\tau$

Where $R_{VI}(\tau) = E[V(t)I(t+\tau)]$ is the * cross - correllation function of V and I.

Calculating Total Noise

If the generator just has thermal noise,

$$\tilde{S}_{E_{N,gen}} = 4kTR_{gen}$$



Represent the combination of amplifier voltage and current noise by a single source

$$E_{Total} = E_N + I_N \cdot Z_{gen}$$

We can now calculate the spectral density of this total noise :

$$\tilde{S}_{E_{n,total,amplifier}} = \| Z_g \|^2 \tilde{S}_{I_n} + 2\,\mathrm{Re}\left\{\tilde{S}_{E_n I_n} Z_g^*\right\}$$

$$= \| Z_g \|^2 \tilde{S}_{I_n} + 2\,\mathrm{Re}\left\{\tilde{S}_{E_n I_n} \left(R_{gen} - jX_{gen}\right)\right\}$$

Signal / Noise Ratio of Generator

V_{signal}, $E_{n,total}$ and $E_{n,gen}$ are in series and see the same load impedance.

The ratios of powers delivered by these will not depend upon the load.

Therefore consider the aviable noise powers.

The signal power available from the generator is $P_{signal,available} = V_{signal,RMS}^2 / 4R_{gen}$

If we consider a narrow bandwidth between $(f_{signal} - \Delta f / 2)$ and $(f_{signal} + \Delta f / 2)$,

then the available noise power from $E_{N,gen}$ is

$$P_{noise,available,generator} = E[E_{n,gen}^2] = \tilde{S}(jf) \cdot \Delta f / 4R_{gen}$$

The signal/noise ratio of the generator is then

$$\text{SNR} = \frac{P_{signal,available}}{P_{noise,available,generator}} = \frac{V_{signal,RMS}^2 / 4R_{gen}}{\tilde{S}_{E_{n,gen}}(jf) \cdot \Delta f / 4R_{gen}} = \frac{V_{signal,RMS}^2 / 4R_{gen}}{kT \cdot \Delta f}$$

Signal / Noise Ratio of Generator+Amplifier

Signal power available from the generator : $P_{signal,available} = V_{signal,RMS}^2 / 4R_{gen}$

Noise power available from generator : $P_{noise,av,gen} = \tilde{S}(jf) \cdot \Delta f / 4R_{gen} = kT \cdot \Delta f$

Noise power available from amplifier : $P_{noise,av,Amp} = \tilde{S}_{E_{n,total,amplifier}}(jf) \cdot \Delta f / 4R_{gen}$

Signal/noise ratio including amplifier noise :

$$SNR = \frac{P_{signal,available}}{P_{noise,avail,gen} + P_{noise,avail,amp}} = \frac{V_{signal,RMS}^2 / 4R_{gen}}{\tilde{S}_{E_{total}}(jf) \cdot \Delta f / 4R_{gen} + \tilde{S}_{E_{n,gen}}(jf) \cdot \Delta f / 4R_{gen}}$$

$$= \frac{V_{signal,RMS}^2 / 4R_{gen}}{\tilde{S}_{E_{total}}(jf) \cdot \Delta f / 4R_{gen} + kT \cdot \Delta f}$$

Noise Figure: Signal / Noise Ratio Degradation

$$\text{Noise figure} = \frac{\text{signal/noise ratio before adding amplifier}}{\text{signal/noise ratio before adding amplifier}}$$

$$\text{Signal/noise ratio before adding amplifier: } SNR = \frac{V_{signal,RMS}^2 / 4R_{gen}}{kT \cdot \Delta f}$$

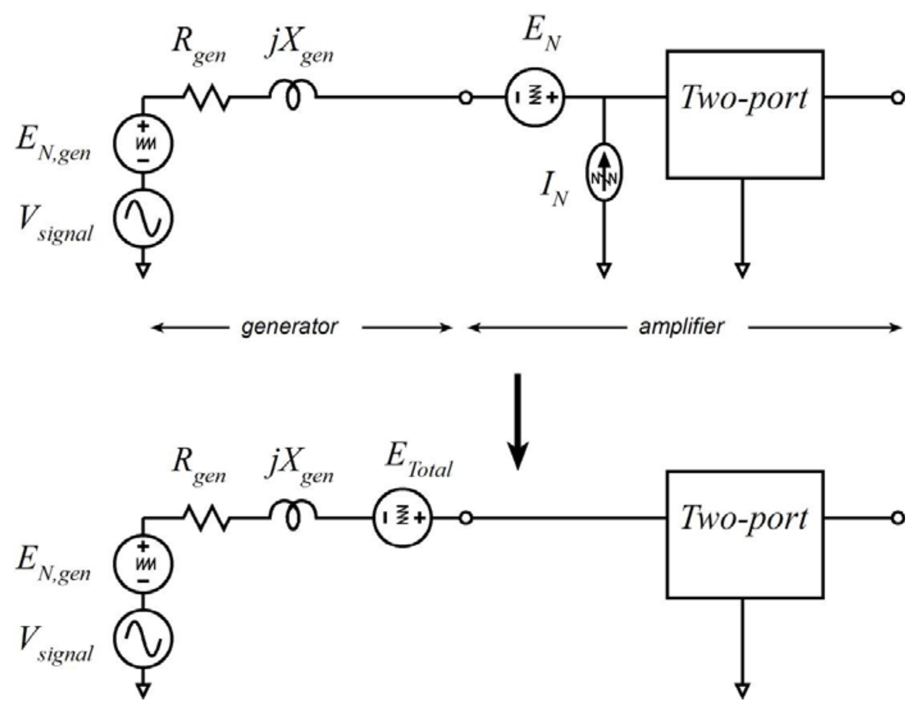
$$\text{Signal/noise ratio after adding amplifier: } SNR = \frac{V_{signal,RMS}^2 / 4R_{gen}}{\tilde{S}_{E_{total}}(jf) \cdot \Delta f / 4R_{gen} + kT \cdot \Delta f}$$

$$\text{Noise figure} = f = \frac{\tilde{S}_{E_{total}}(jf) \cdot \Delta f / 4R_{gen} + kT \cdot \Delta f}{kT \cdot \Delta f}$$

$$\text{Noise figure} = 1 + \frac{\tilde{S}_{E_{total}}(jf) / 4R_{gen}}{kT} = 1 + \frac{\text{amplifier available input noise power}}{kT}$$

Calculating Noise Figure

$$\text{Noise figure} = 1 + \frac{\tilde{S}_{E_{total}}(jf)/4R_{gen}}{kT}$$



We also know that:

$$\tilde{S}_{E_{n,total,amplifier}} = \| Z_g \|^2 \tilde{S}_{I_n} + 2\,\mathrm{Re}\left\{ \tilde{S}_{E_n I_n} Z_g^* \right\}$$

We can calculate from this an expression for noise figure:

$$F = 1 + \frac{S_{E_n} + |Z_s|^2 S_{I_n} + 2 \cdot \mathrm{Re}\left(Z_s^* S_{E_n I_n} \right)}{4kTR_{gen}}$$

Minimum Noise Figure

Noise figure varies as a function of $Z_{gen} = R_{gen} + jX_{gen}$:

$$F = 1 + \frac{S_{E_n} + |Z_s|^2 S_{I_n} + 2 \cdot \mathrm{Re}\left(Z_s^* S_{E_n I_n}\right)}{4kTR_{gen}}$$

After some calculus, we can find a mimimum noise figure and a generator impedance which gives us this minimum :

$$F_{\min} = 1 + \frac{1}{4kT}\left[2\sqrt{S_{E_n} S_{I_n} - \left(\mathrm{Im}\left[S_{E_n I_n}\right]\right)^2} + 2\,\mathrm{Re}\left[S_{E_n I_n}\right]\right]$$

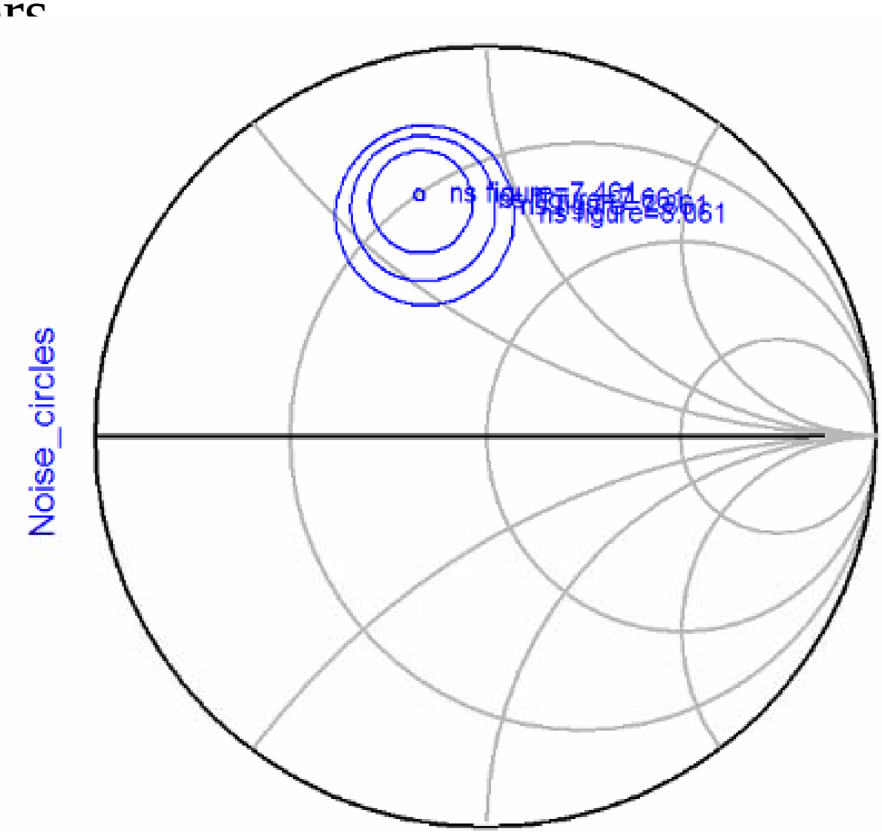
$$Z_{opt} = R_{opt} + jX_{opt} = \sqrt{\frac{S_{E_n}}{S_{I_n}} - \left(\frac{\mathrm{Im}\left[S_{E_n I_n}\right]}{S_{I_n}}\right)^2} - j\,\frac{\mathrm{Im}\left[S_{E_n I_n}\right]}{S_{I_n}}$$

Points to remember are (1) F varies with Z_{gen}, hence there is an optimum Z_{gen} (2) which gives a minimum F (3).

Noise Figure in Wave Notation

Written instead in terms of wave parameters

$$F = F_{\min} + \frac{4r_n \cdot \left\| \Gamma_s - \Gamma_{opt} \right\|^2}{\left[1 - \left\| \Gamma_s \right\|^2\right] \cdot \left[1 - \Gamma_{opt}\right]^2}$$



These describe contous in the Γ_s – plane of constant noise figure : "noise figure circles", i.e. a description of the variation of noise figure with source reflection coefficient.

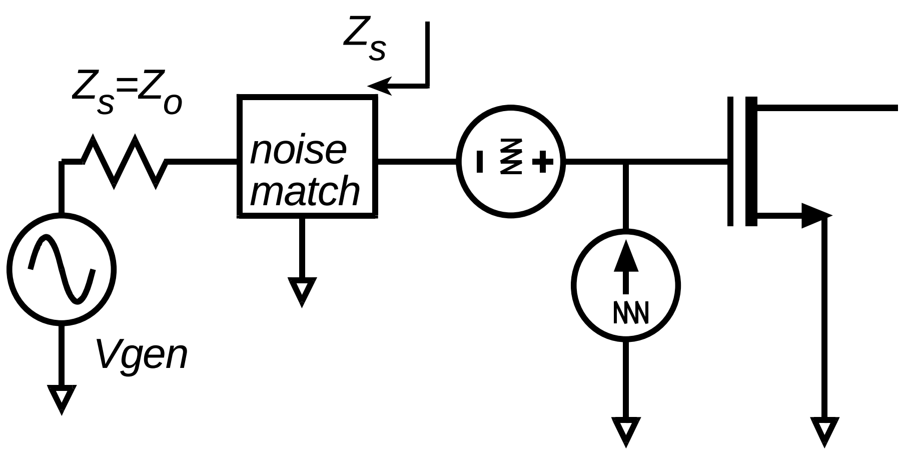
Low-Noise Amplifier Design

Design steps are
1) in-band stabilization: this is best done at output port to avoid degrading noise
2) input tuning for F_{min}
3) output tuning (match)
4) out-of-band stabilization

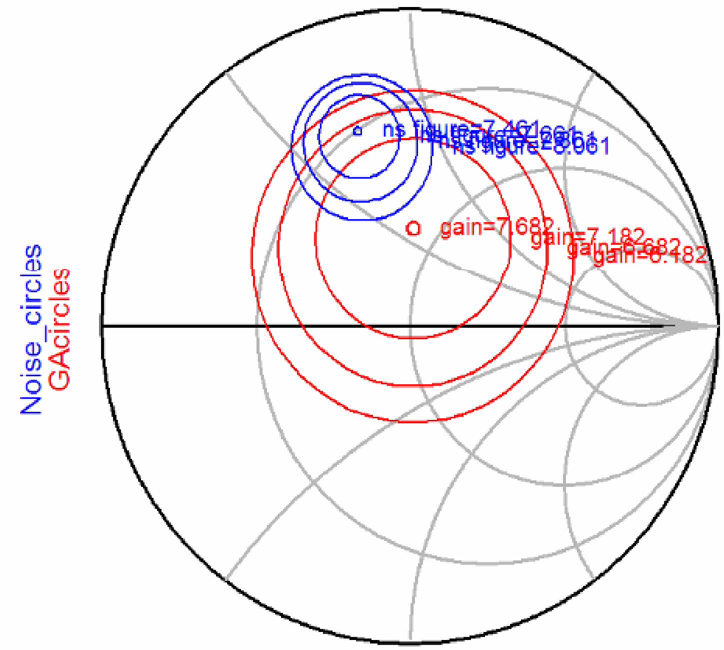
Note that tuning for minimum noise figure requires a *mismatch* on the amplifier input; amplifier gain therefore must lie below the transistor MAG/MSG.

Note that tuning for minimum noise figure implies that amplifier input is mismatched: input reflection coefficient is therefore not zero !

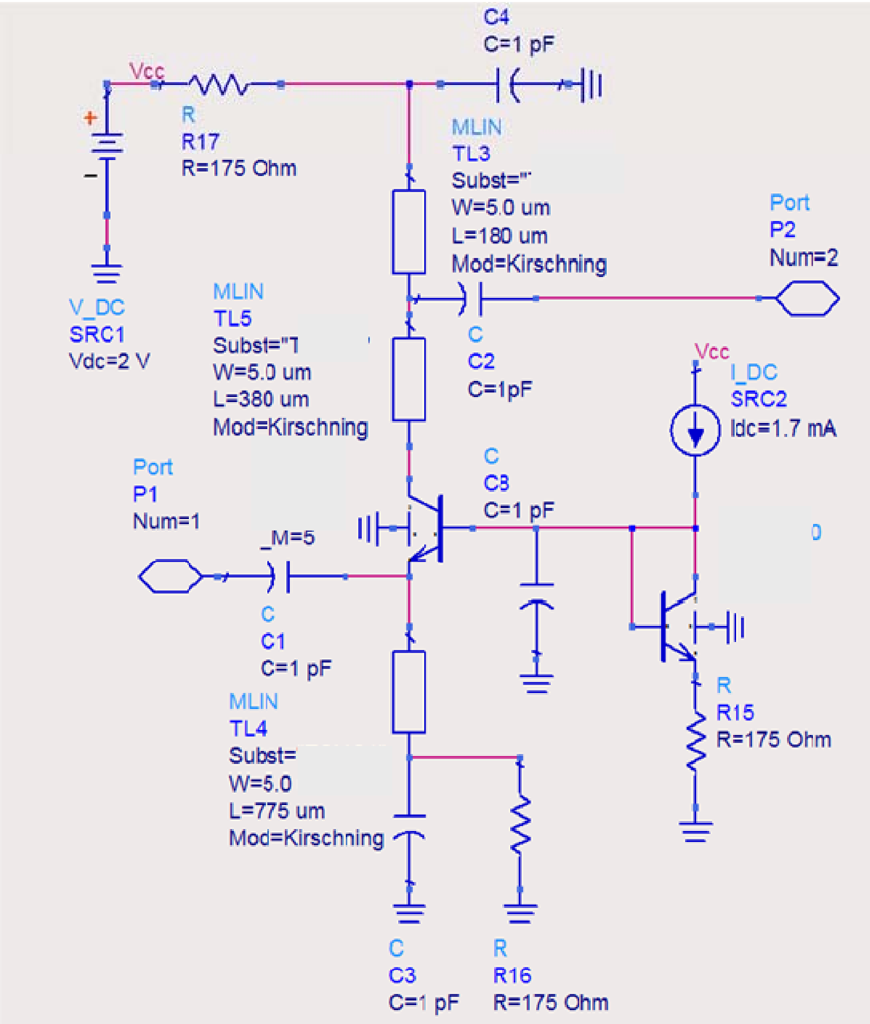
Discrepancy in input noise-match & gain-match can be reduced by adding source inductance

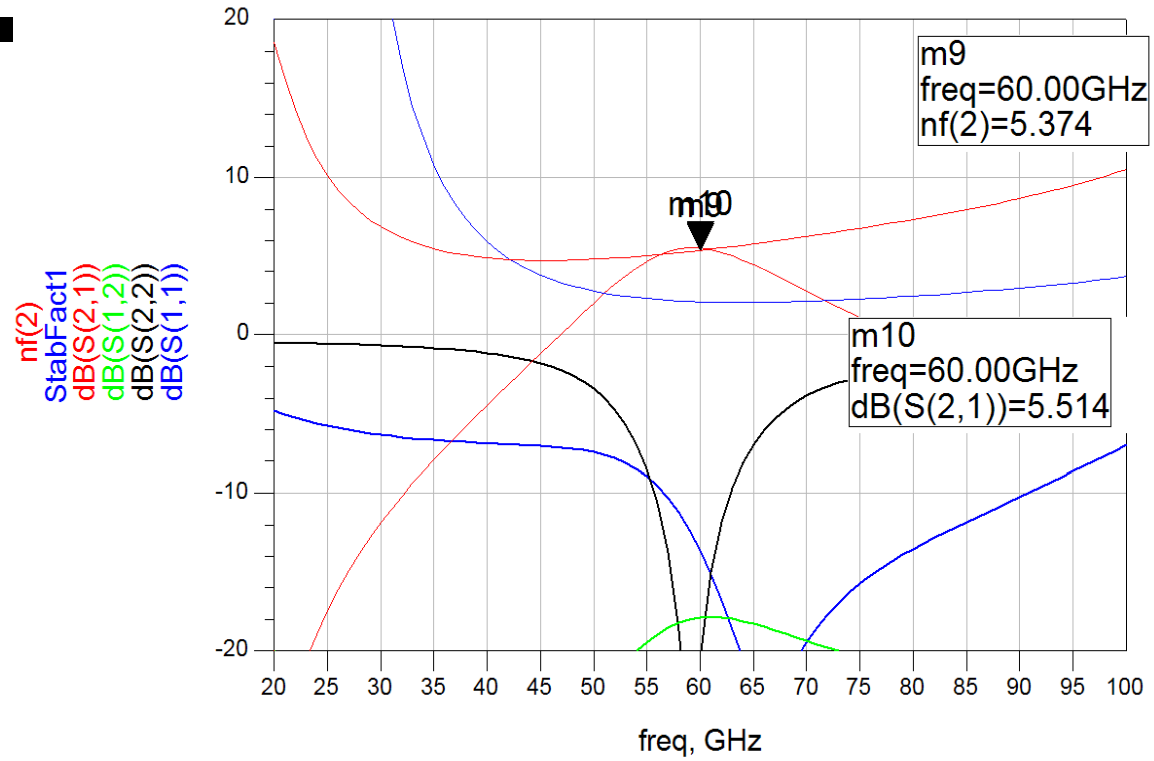


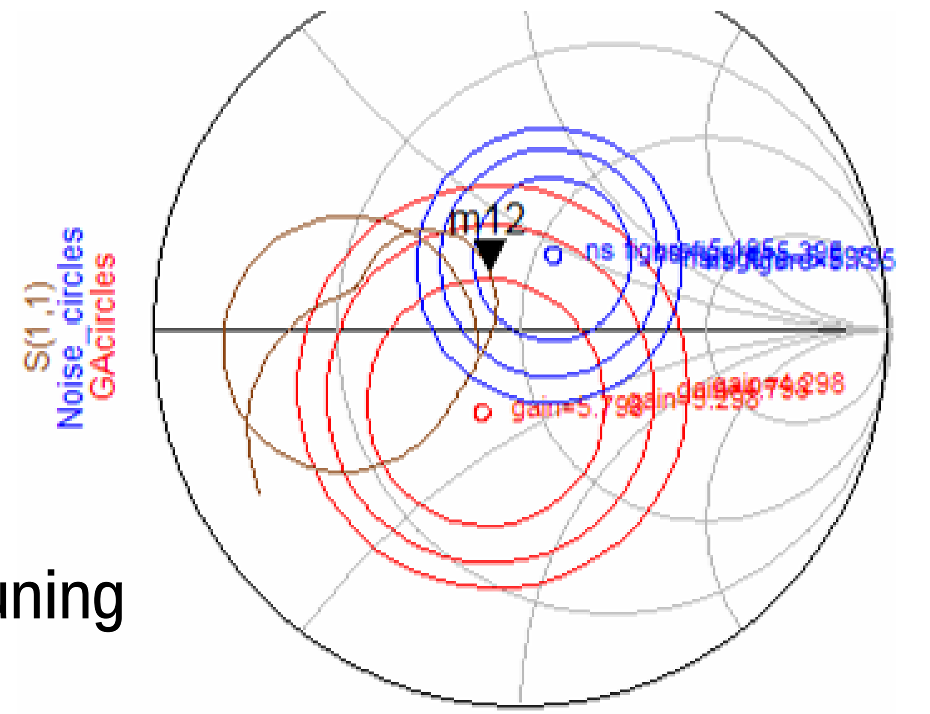
Noise and Available Gain Circles



Example LNA Design: 60 GHz, 130 nm SiGe BJT

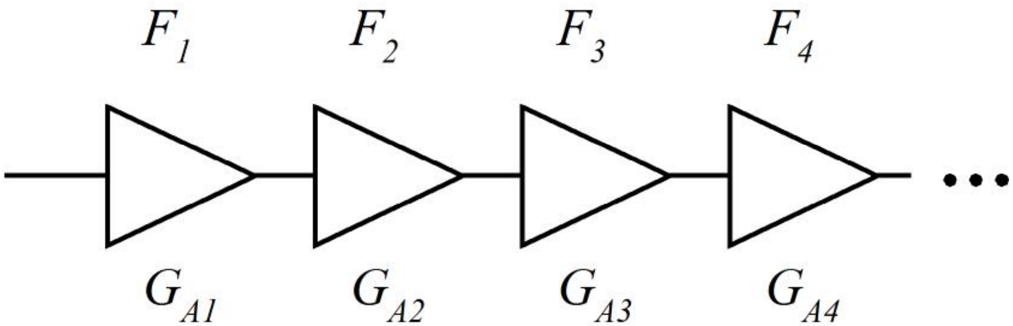






gain & noise circles after input matching
note compromise between gain & noise tuning

Noise Figure of Cascaded Stages



$$F_{cascade} = F_1 + \frac{F_2 - 1}{G_{A1}} + \frac{F_3 - 1}{G_{A1} G_{A2}} + \frac{F_4 - 1}{G_{A1} G_{A2} G_{A3}} + \ldots$$

This is the Friis noise figure formula.

G_{Ai} are the available power gains.

The relationship applies whether or not inter - stages are matched.