# *ECE 145A / 218 C, notes set 1: Transmission Line Properties and Analysis*

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# Transmission Line Analysis

Geometries

Characteristic Impedances

Time Domain Analysis

Lattice Diagrams

Frequency Domain analysis

Reflection coefficients

Movement of Refernece Plane

Impedance vs Position

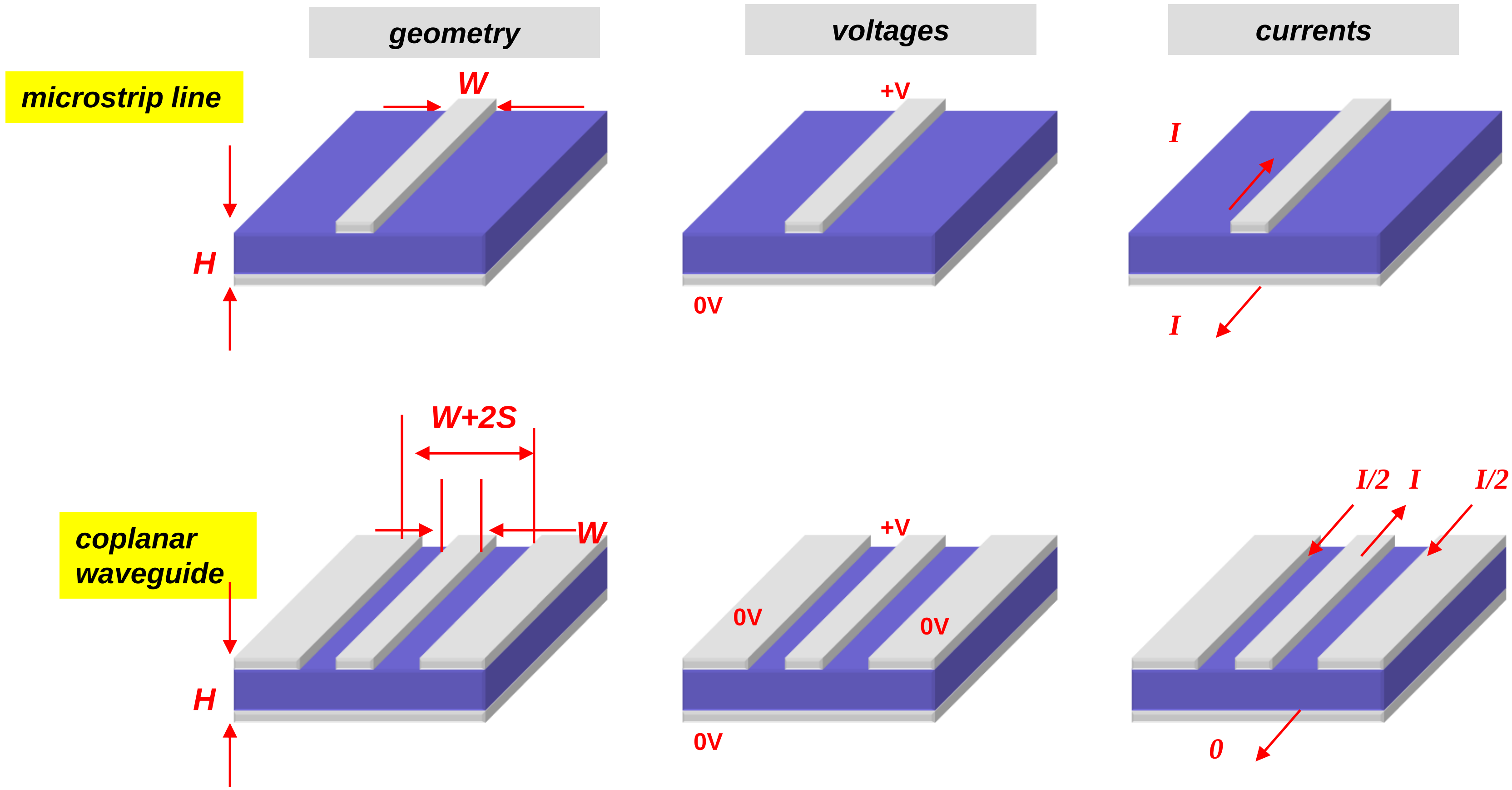
Smith Chart

Standing Waves

Solving wave equations quickly

# types of transmission lines

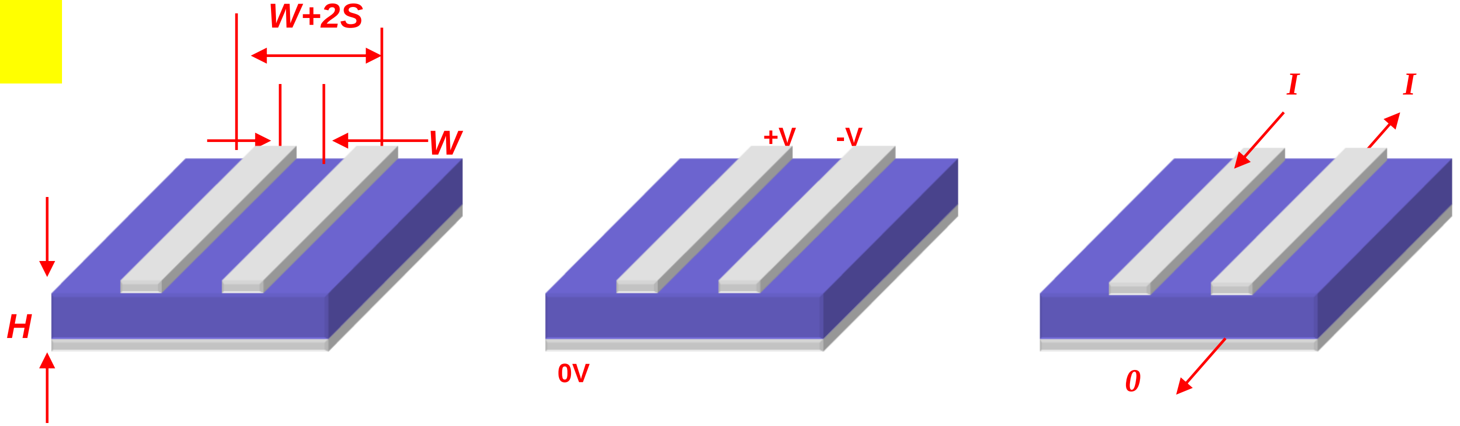
# Transmission Lines for On-Wafer Wiring



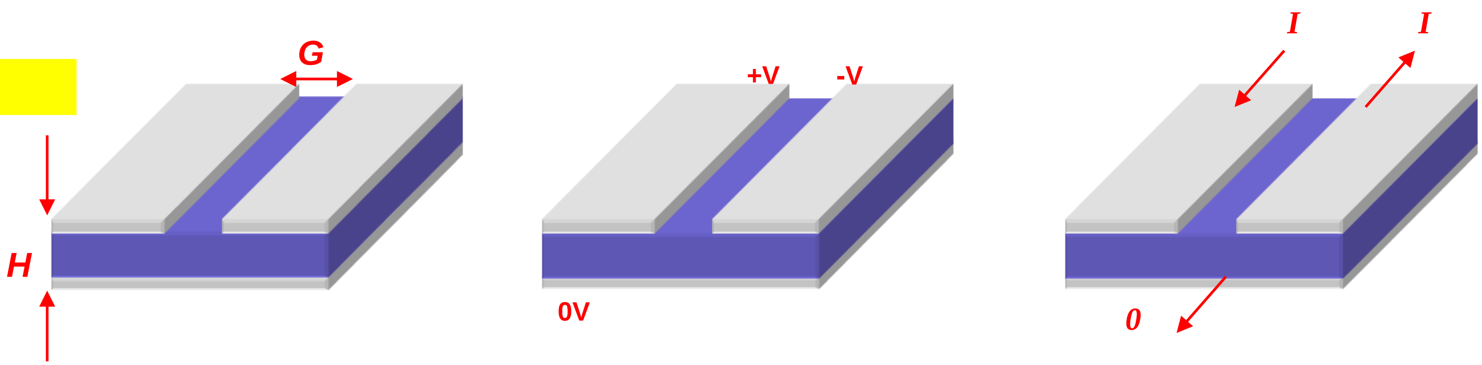
# Transmission Lines for On-Wafer Wiring

**geometry** | **voltages** | **currents**

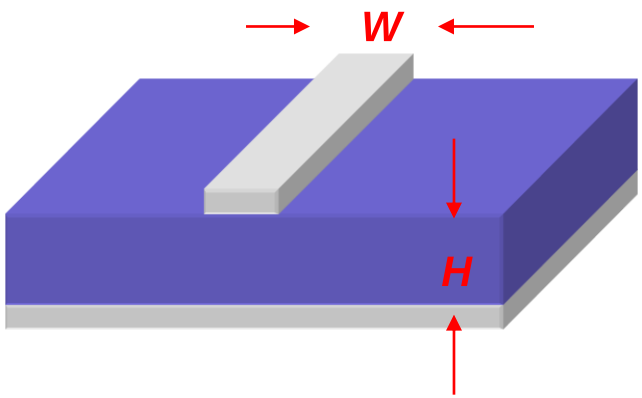
***coplanar strips***



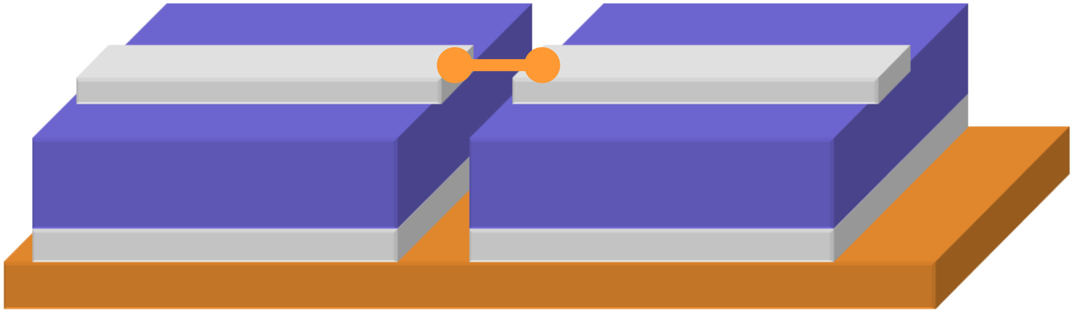

***slotline***



# Substrate Microstrip Line

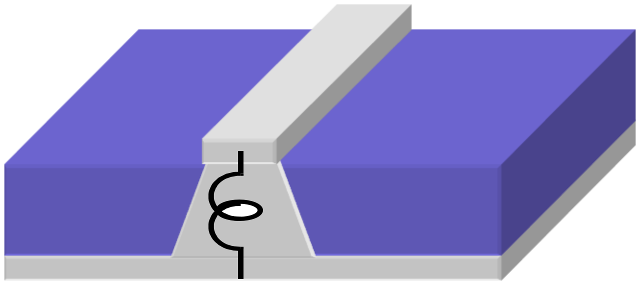



Dominant Transmission medium in III-V microwave & mm-wave ICs



Key advantage: IC interconnects have very low ground-lead inductance

Ground-lead inductance:
-leads to ground-bounce
-is Miller-multiplied by IC gain



Key problems:
through-wafer grounding holes (vias)
coupling to TM modes in substrate

Via inductance forces progressively thinner wafers at higher frequencies.

# basic theory
# L, C, Zo, velocity, Gamma

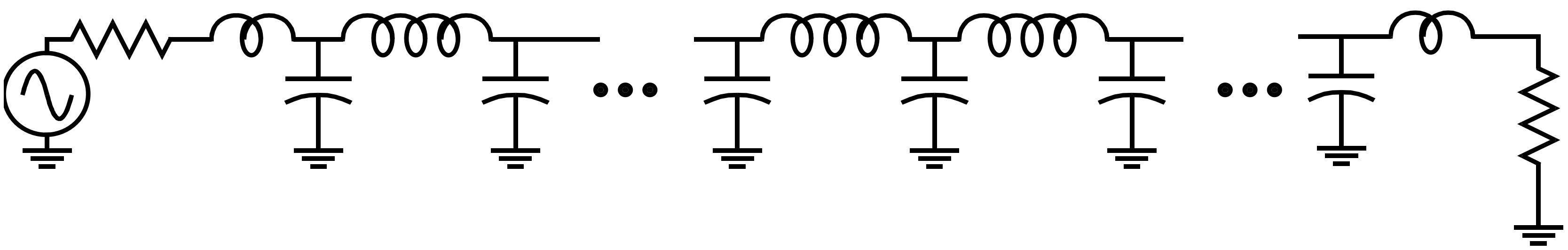
# Transmission Lines

A pair of wires with ***regular spacing, dielectric loading*** along the length.

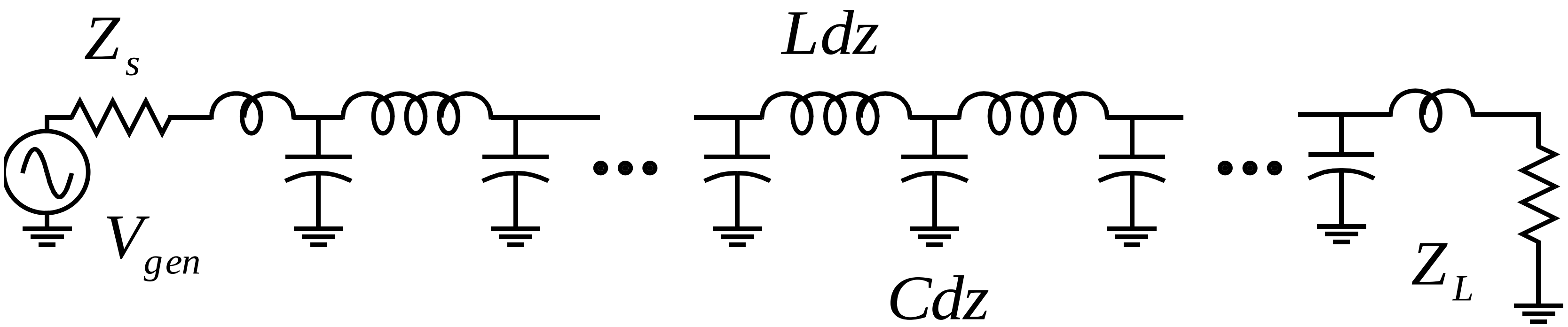
These have inductance per unit length and capacitance per unit length.

Forward and reverse waves propagate.

Reflections will occur if lines are not correctly terminated



# Transmission Lines: Basic Theory




From basic nodal analysis of line :

$(dV/dz) = -L(dI/dt)$ and

$(dI/dz) = -C(dV/dt)$ from which we find

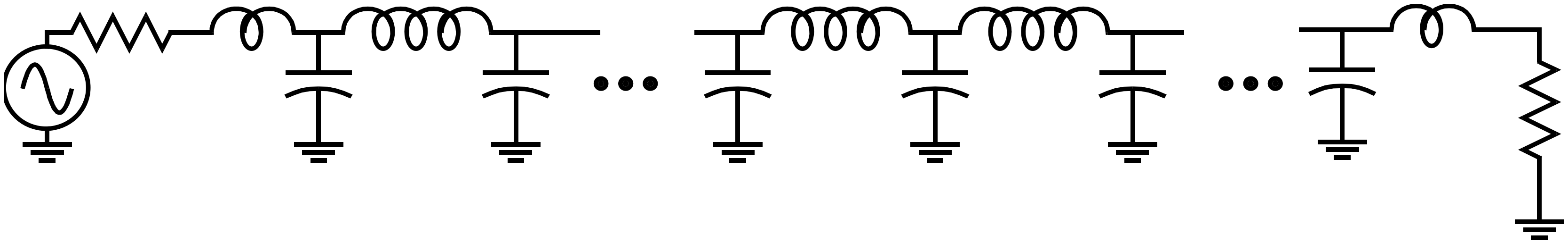
$$V(z,t) = V^+(t - z/v) + V^-(t + z/v)$$

$$I(z,t) = \frac{V^+(t - z/v)}{Z_o} - \frac{V^-(t + z/v)}{Z_o}$$

where

$Z_o = \sqrt{L/C}$ and $v = 1/\sqrt{LC}$

# Forward and Reverse Waves



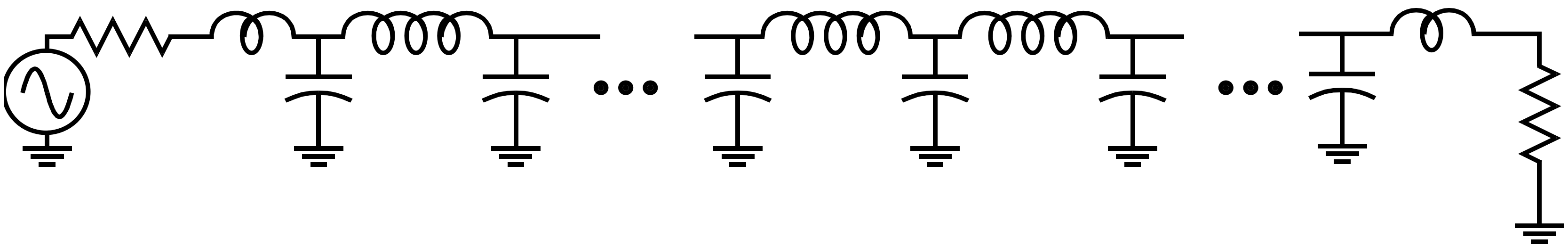
$V^{+}(t-z/v)$ voltage in forward wave

$+V^{-}(t+z/v)$ voltage in reverse wave

$\dfrac{V^{+}(t-z/v)}{Z_o}$ current in forward wave

$-\dfrac{V^{-}(t+z/v)}{Z_o}$ current in reverse wave

# Velocity and Characteristic Impedance



$Z_o = \sqrt{L/C}$ and $v = 1/\sqrt{LC}$

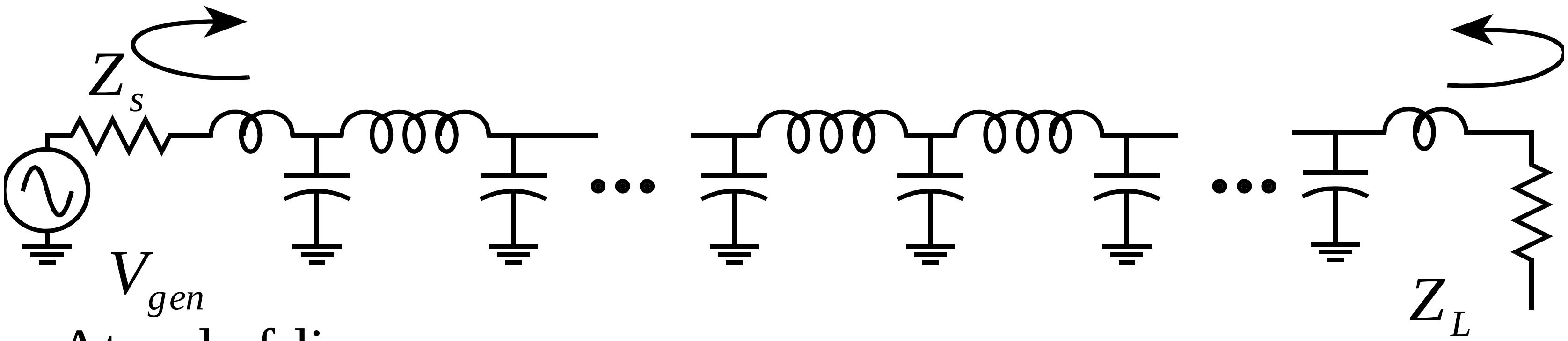
$L$ and $C$ are here quantities per unit length.

$v = c/\sqrt{\varepsilon_{r,eff}}$

where $c$ is the speed of light and

$\varepsilon_{r,eff}$ is the effective dielectric constant of the line

# Reflections




At end of line :

$$V^- = \Gamma_l V^+ \quad \text{where } \Gamma_l = \frac{(Z_l/Z_o)-1}{(Z_l/Z_o)+1}$$

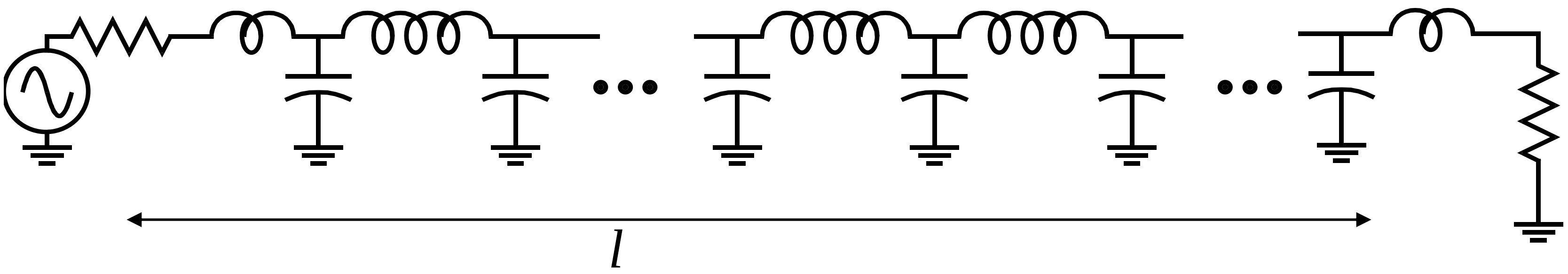
At beginning of line :

$$V^+ = \Gamma_s V^- + T_s V_{gen} \quad \text{where } \Gamma_s = \frac{(Z_s/Z_o)-1}{(Z_s/Z_o)+1}$$

$$\text{and } T_s = \frac{Z_o}{Z_o + Z_s}$$

Need good terminations to prevent line reflections and ringing

# Total inductance & capacitance in a length of line




If total line length is $l_{length}$

Then total capacitance in that length is

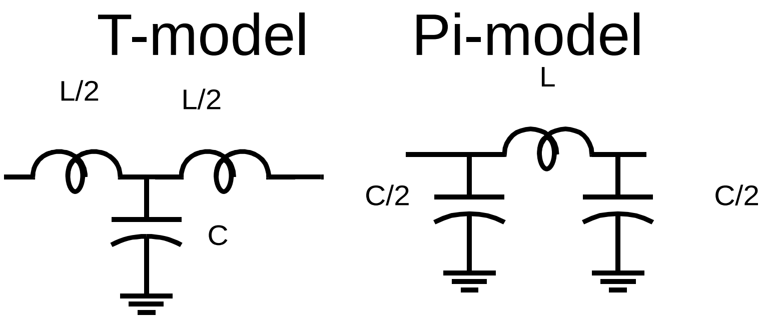
$$C_{length} = \frac{\tau}{Z_o}$$

and total inductance in that length is

$$L_{length} = \tau Z_o$$

where $\tau = l_{length} / v =$ "speed of light delay" on the line

# Lumped models of very short transmission lines




If total line length $l_{length}$ is much less than a wavelength

or total line delay $\tau = l_{length}/v$ is much less than $1/f_{signal}$

or total line delay $\tau$ is much less than pulse risetime

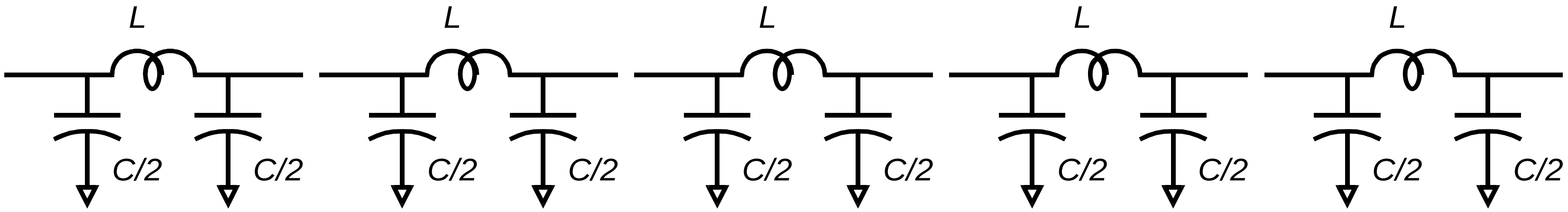
then the line can be approximated as a T or $\pi$ section

$$C_{length} = \frac{\tau}{Z_o}$$

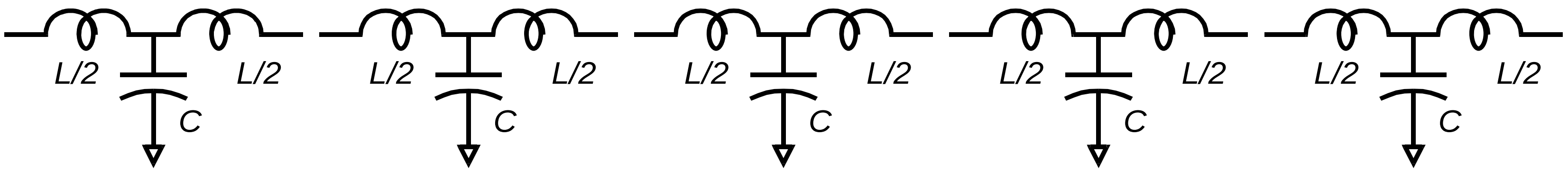
$$L_{length} = \tau Z_o$$

# Ladder models of moderately short transmission lines

Pi-model synthesis




T-model synthesis



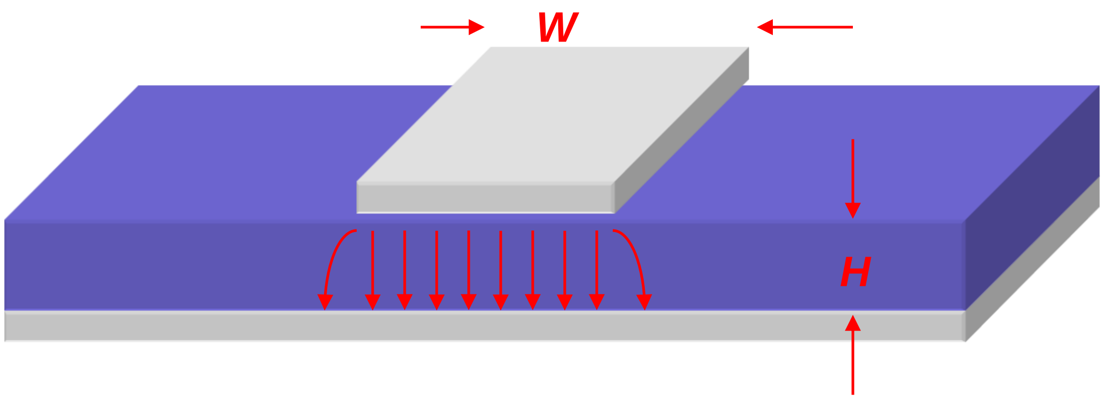

Clearly, we can break a line of any length into sections of length $l_{line}$ such that $\tau_{\text{line}} = l_{line} / v$ is much less than a signal period.

In this fashion a transmission-line can be modelled by an LC filter.

This is a frequent substitution in circuit simulations

# Microstrip Lines

# Microstrip Line: Approximate Properties (1)



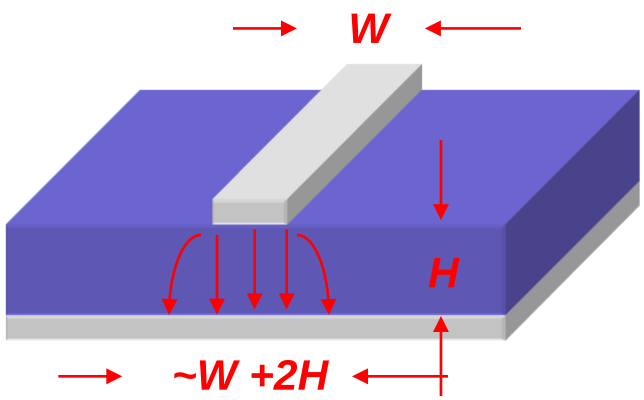

Wide line → field mostly in dielectric. This gives :

$v = c/\varepsilon_r^{1/2}$, where $c = 1/\sqrt{\mu\varepsilon}$ is the speed of light

$Z_0 = \eta_0 H/\varepsilon_r^{1/2} W$, where $\eta_0 = \sqrt{\dfrac{\mu}{\varepsilon}}$ is the free space wave impedance

(note: wide lines have problems)

# Microstrip Line: Approximate Properties (2)




If the line is narrower, hand analysis is only approximate

Effective width $\approx W + 2H$

$Z_0 \cong \eta_0 H / \varepsilon_r^{1/2}(W + 2H)$ only very appoximately

$v = c / \varepsilon_{r,eff}^{1/2}$

$\varepsilon_{r,eff}$ lies somewhere between that of air and of the dielectric,

depending upon what proportion of the field is in air.

# Lines in
# Time Domain

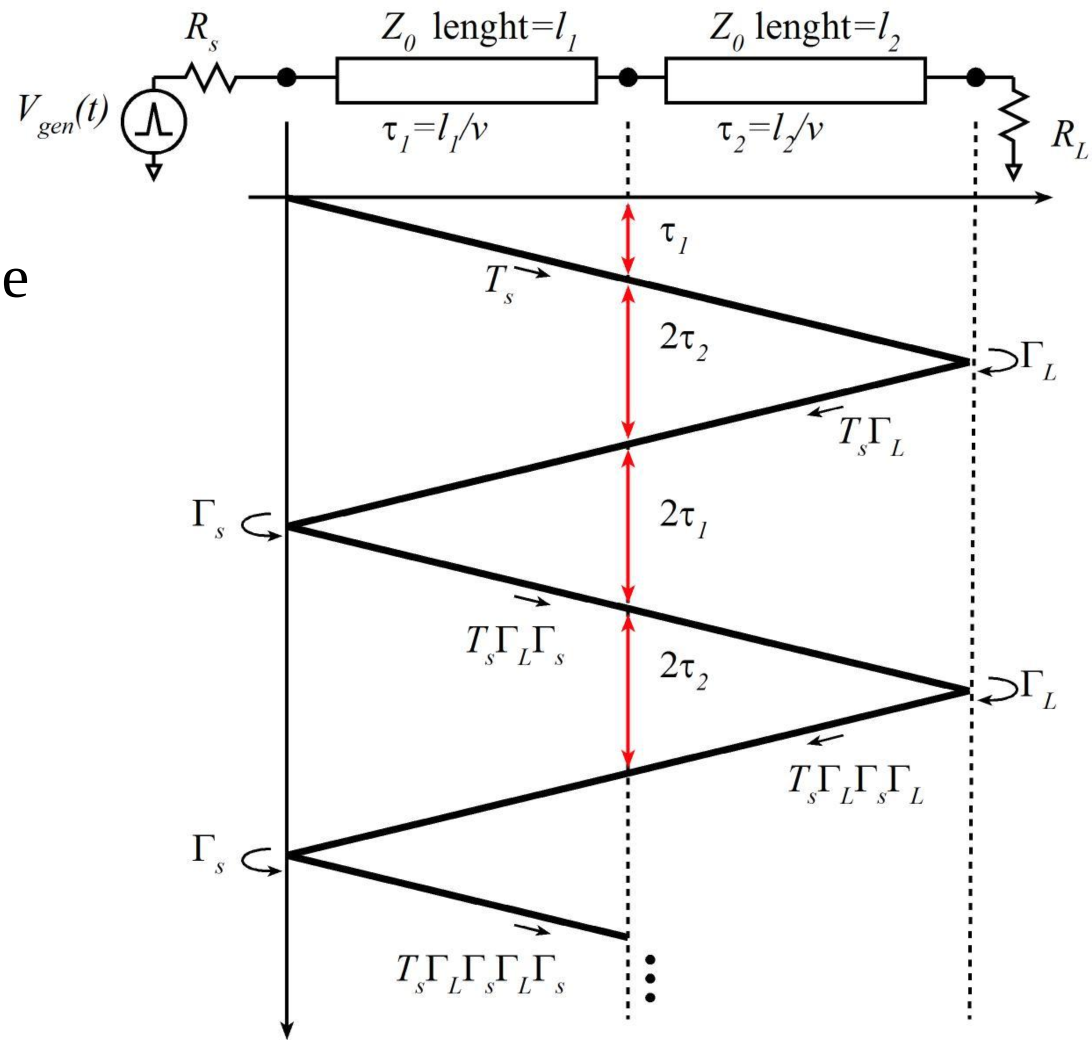
# Lattice Diagrams = Echo Diagrams

First:

Analyze for impulse response

Then:

Use convolution to find general reponse.



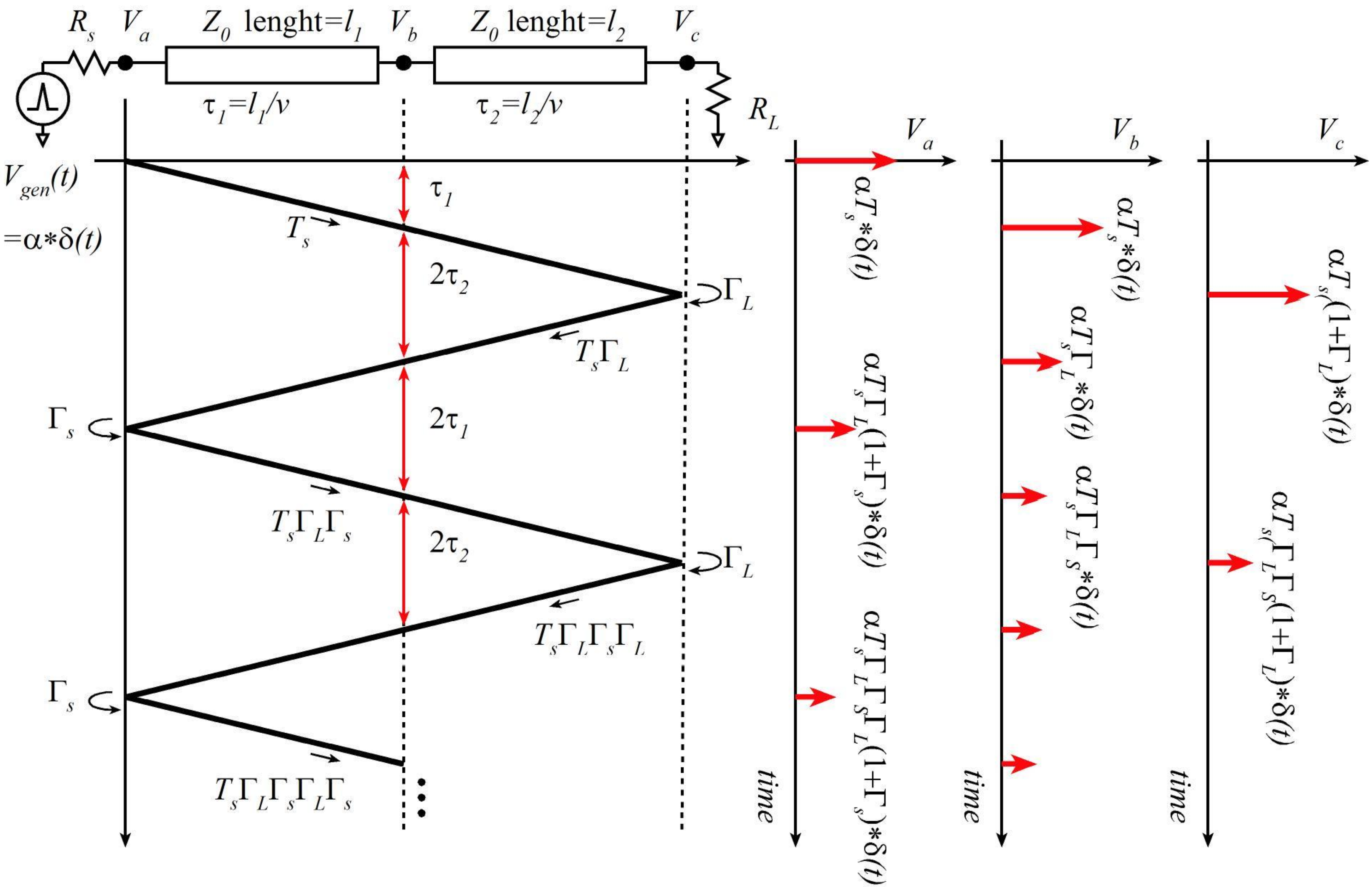

Recall : At end of line :

$$V^- = \Gamma_L V^+ \quad \text{where } \Gamma_L = \frac{(R_L/Z_o)-1}{(R_L/Z_o)+1}$$

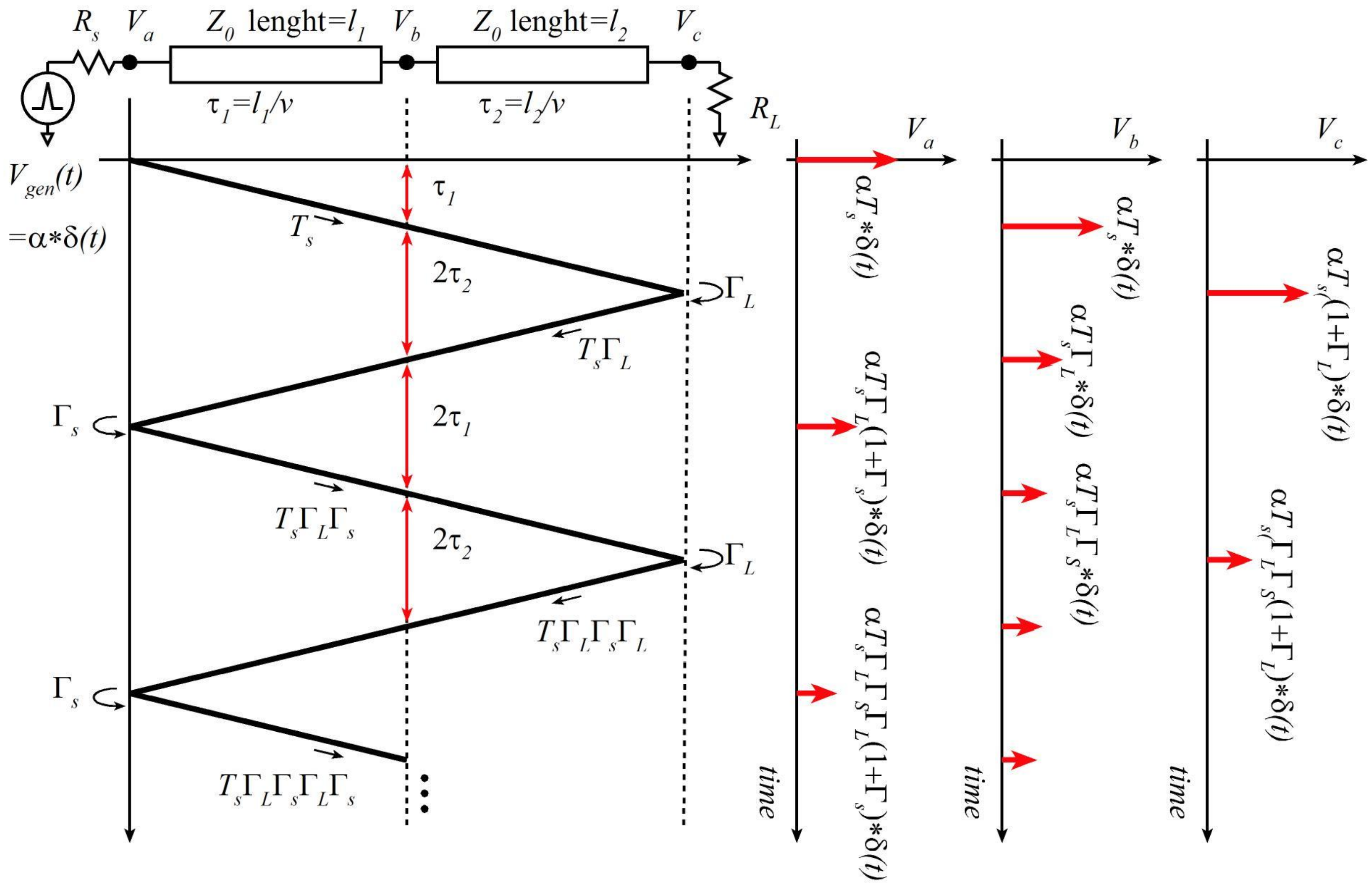
At beginning of line :

$$V^+ = \Gamma_s V^- + T_s V_{gen} \quad \text{where } \Gamma_s = \frac{(R_s/Z_o)-1}{(R_s/Z_o)+1} \text{ and } T_s = \frac{Z_o}{Z_o + R_s}$$

# Lattice Diagrams = Echo Diagrams



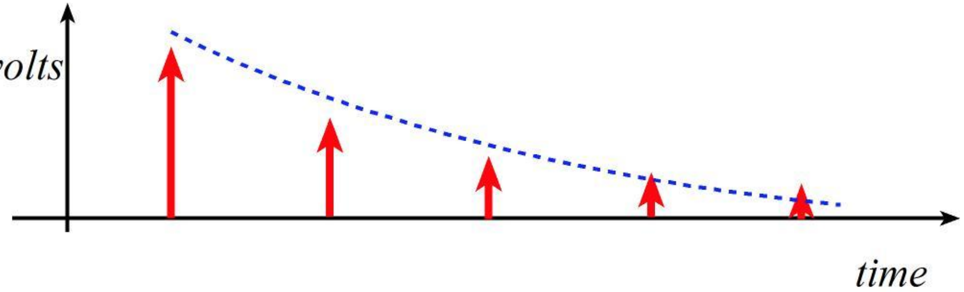
# Lattice Diagrams = Echo Diagrams



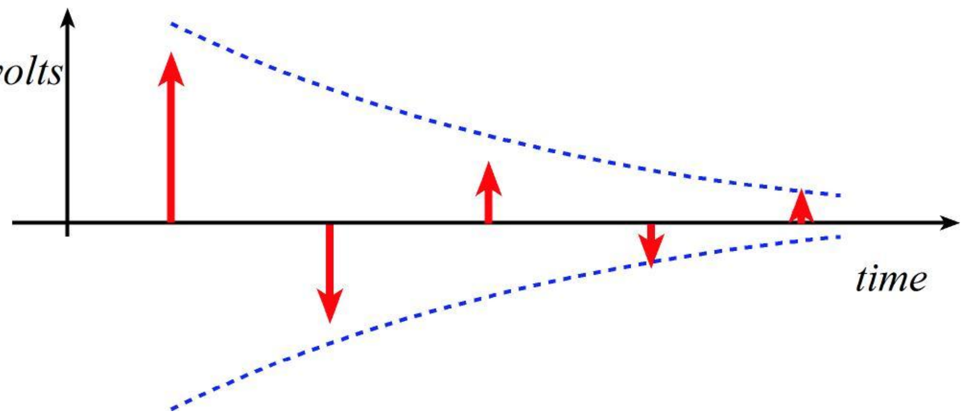

Now please consider how the waveforms would change if the generator were a step-function.

# Repeated Reflections→ Ringing or Exponential Decay

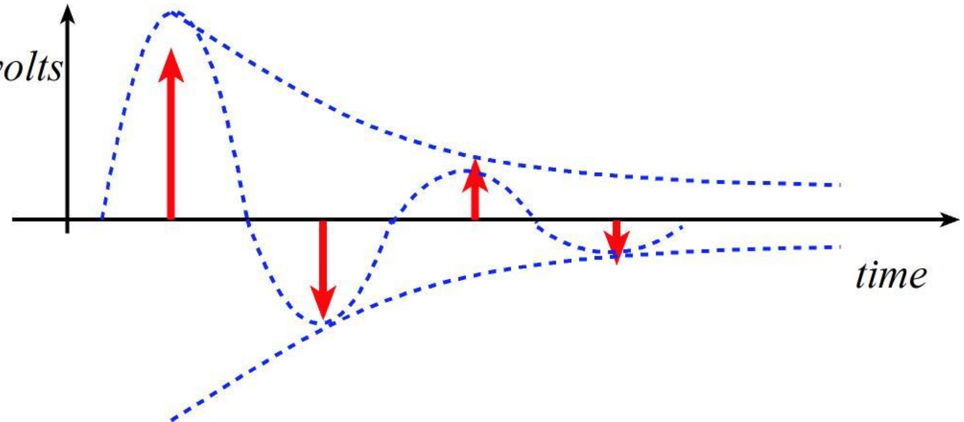
If $\Gamma_L \Gamma_s$ is positive, pulse responses decay geometrically (exponentially)



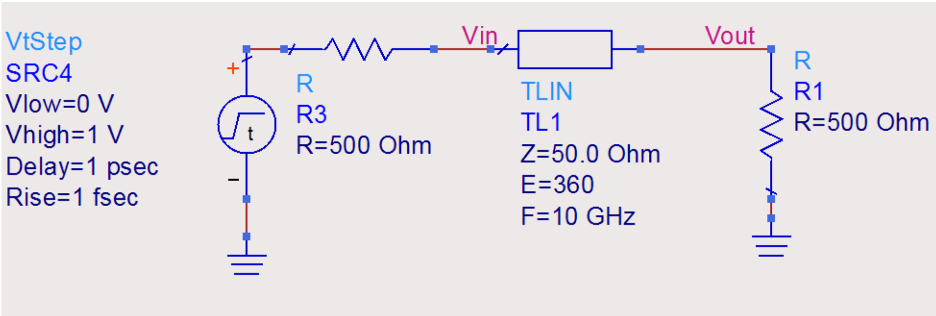

If $\Gamma_L \Gamma_s$ is negative, pulse responses also alternate in sign --ringing.

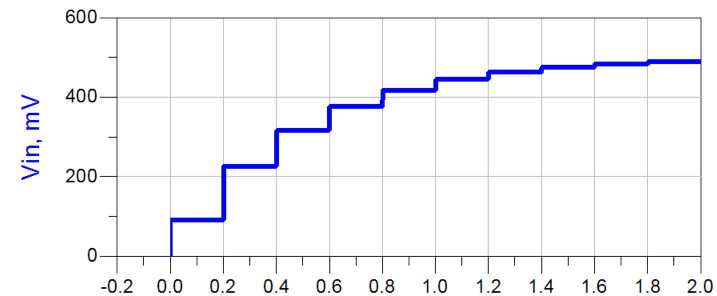


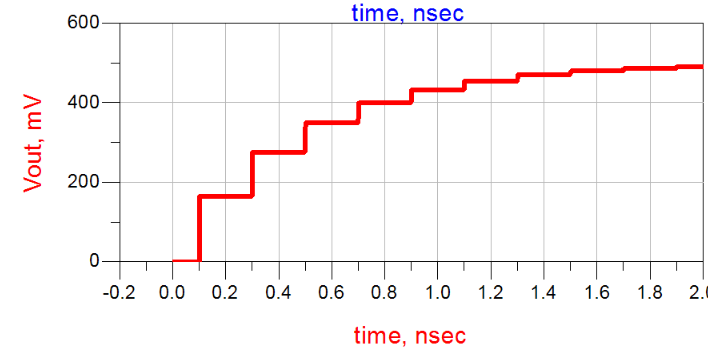

Behavior appears very close to RLC ringing. Why ?



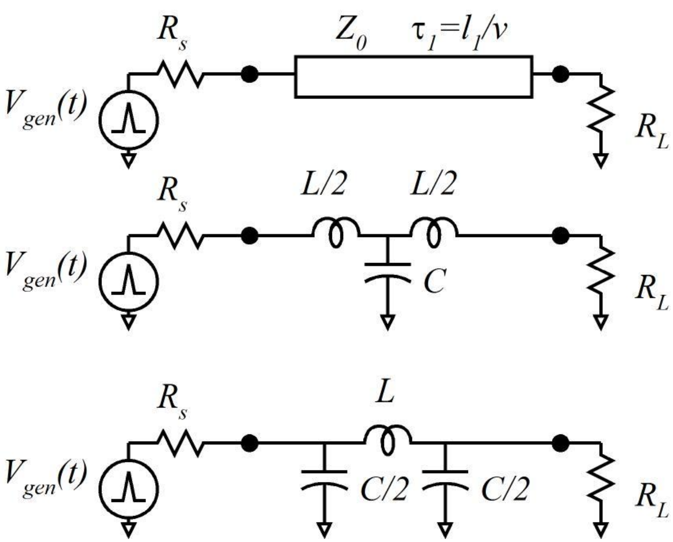
# Time-Domain Analysis



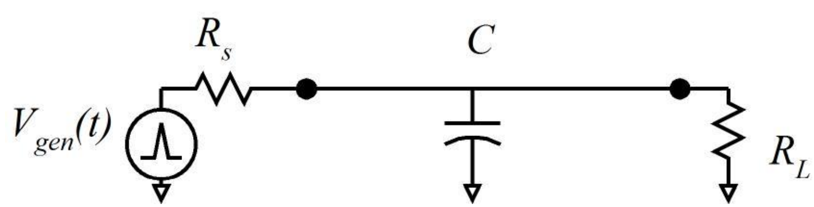




$$L = Z_0 \tau \, , C = \tau / Z_0$$




Approximate model

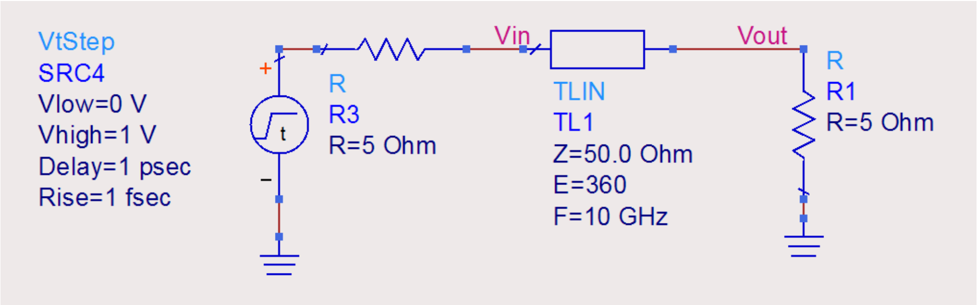


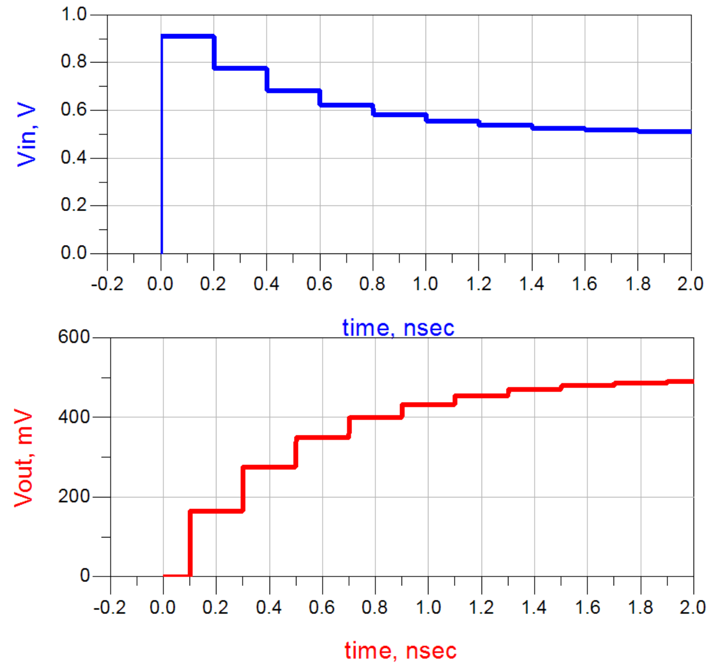

$$L/(R_L + R_s) << (R_L \parallel R_s)C$$

$\rightarrow$ neglect inductor

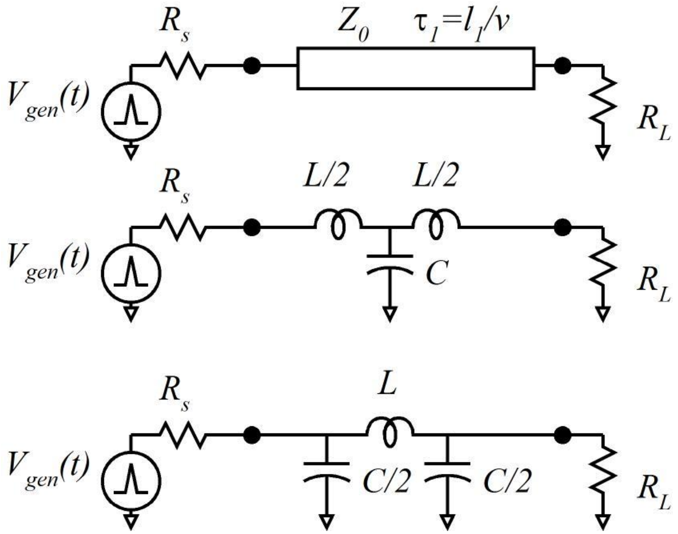
RC circuit $\rightarrow$ charging.

# Time-Domain Analysis

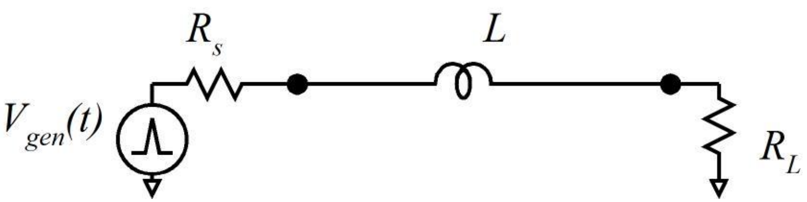



$$L = Z_0 \tau\,,\, C = \tau / Z_0$$




Approximate model

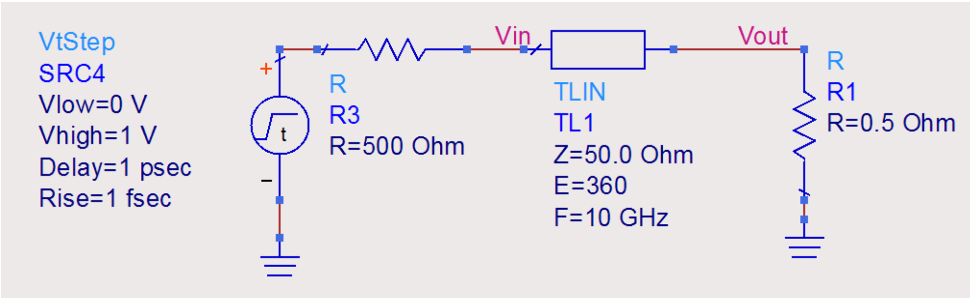


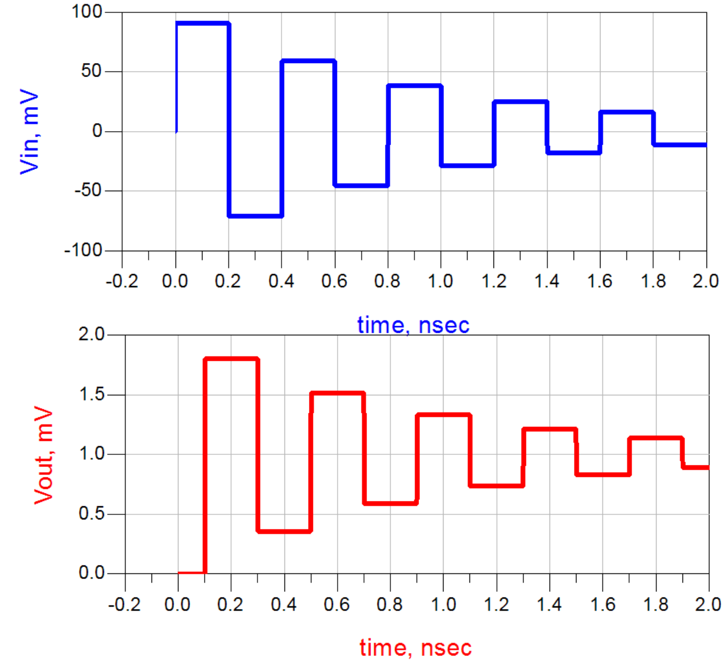

$$L/(R_L + R_s) >> (R_L \parallel R_s) C$$

$\rightarrow$ neglect capacitor

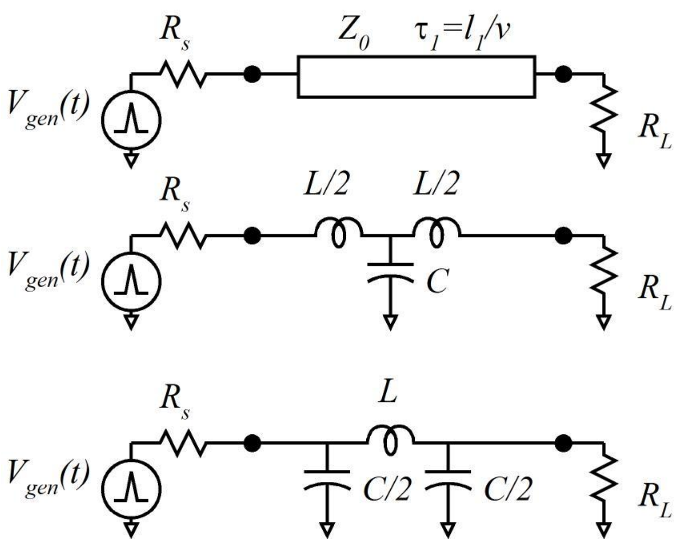
RL circuit $\rightarrow$ charging.

# Time-Domain Analysis

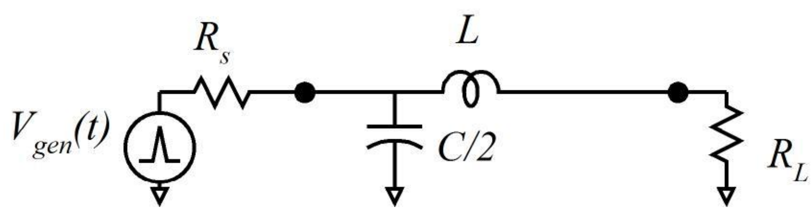



$$L = Z_0 \tau \, , C = \tau / Z_0$$




Approximate model




$$R_L C / 2 << R_S C / 2$$

$\rightarrow$ neglect 2nd capacitor

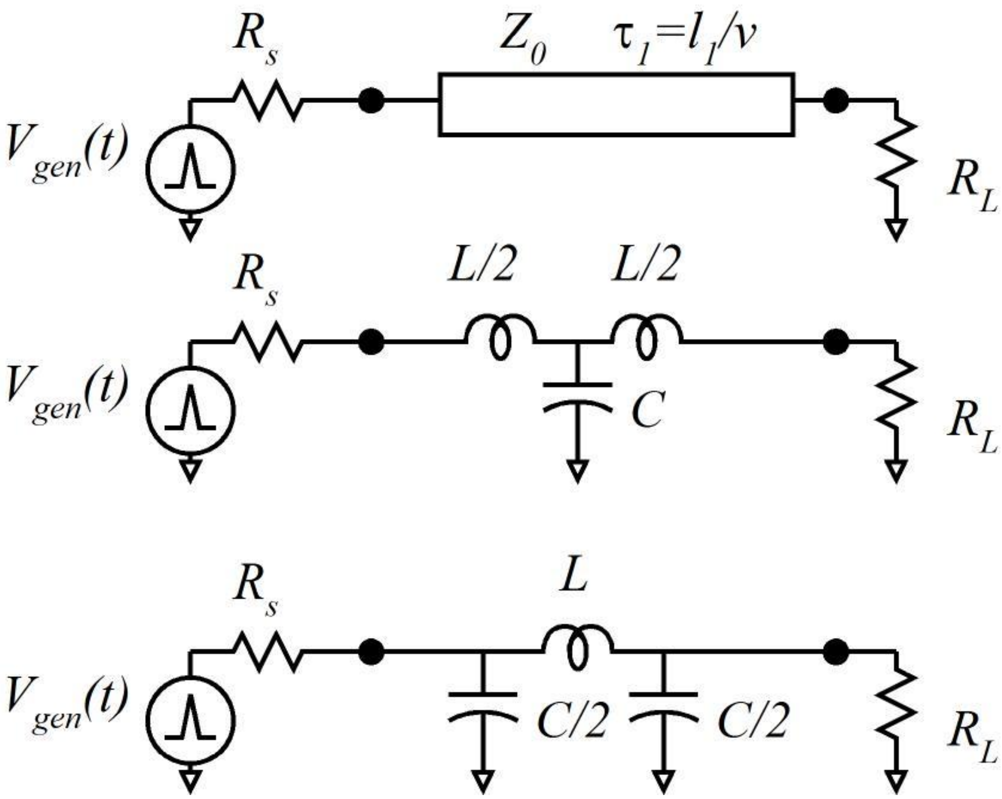
RLC circuit $\rightarrow$ ringing

# L and C are Limiting Cases of High-$Z_0$, low-$Z_0$ lines

$$L = Z_0 \tau \,,\, C = \tau / Z_0$$

High - $Z_0$ line :
large $L$, small $C$.
$\rightarrow$ approximately
an inductor

Low - $Z_0$ line :
large $C$, small $L$.
$\rightarrow$ approximately
a capacitor.



# Lines in
# Frequency Domain

# Line Analysis in Frequency Domain→ Smith Chart

Time-domain analysis:

intuititive and clear: pulses bouncing back and forth.

very difficult with reactive (L, C) load or generator impedances

Frequency-domain analysis:

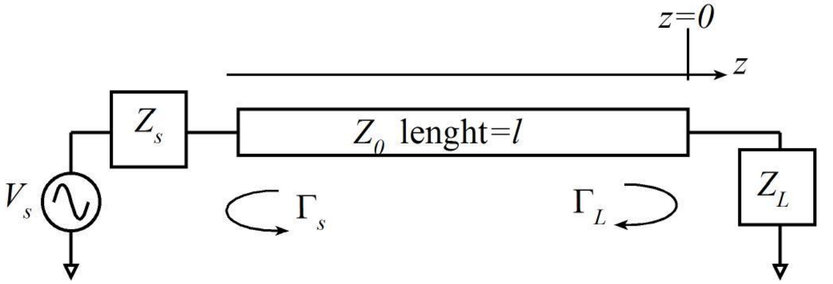
less intuititive.

easy with reactive (L, C) load or generator impedances

→ (1) standing waves

→ (2) Smith chart

# Line Analysis in Frequency Domain: Phase Constant β




Phasor notation: $V_s(t) = \mathrm{Re}\left[V_0 e^{j\omega t}\right]$, where is $V_0 = \| V_o \| e^{j\theta_o}$ is complex.

$\rightarrow V_s(t) = V_0 \cos(\omega t + \theta_o)$

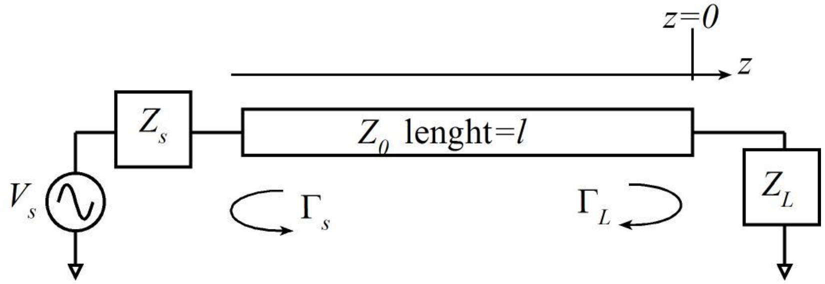
On a transmission line, waves travel as $V^+(t - z/v), V^-(t + z/v)$.

For a cosinusoidal wave traveling at velocity $v$,

$\cos(\omega(t \pm z/v) + \theta) = \cos(\omega t \pm \omega z/v + \theta) = \cos(\omega t \pm \beta z + \theta)$.

$\beta = \omega / v = 2\pi / \lambda$ is the phase propagation constant.

# Line Analysis: Exponential Waves




Because $V_0\cos(\omega t+\theta_o)=\mathrm{Re}\left[V_0 e^{j\omega t}\right]$, sinusiodal waves are written implicitly as $V_0 e^{j\omega t}$.

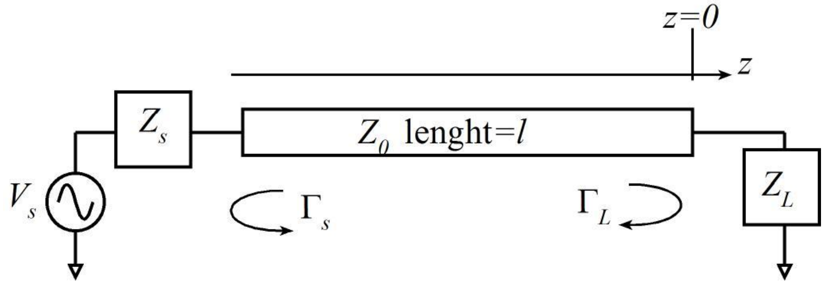
Exponential waves propagating in the positive z - direction :

$$V^+ e^{j\omega(t-z/v)} = V^+ e^{j\omega t - j\omega z/v} = V^+ e^{j\omega t} e^{-j\beta z}$$

Exponential waves propagating in the negative z - direction :

$$V^- e^{j\omega(t+z/v)} = V^- e^{j\omega t + j\omega z/v} = V^- e^{j\omega t} e^{+j\beta z}$$

# Voltages on a Transmission Line



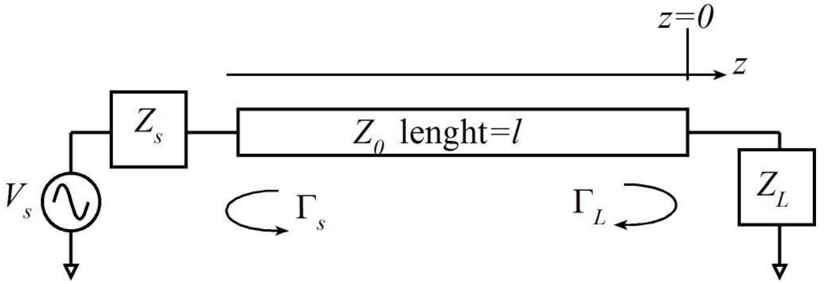

Voltage on line : $V(z,t) = \mathrm{Re}[V(z)e^{j\omega t}]$

Working with the phasor $V(z)$ makes $e^{j\omega t}$ time dependence implicit.

Phasor voltage on the line :

$$V(z) = V^+(z) + V^-(z)$$
$$= V^+(0)e^{-j\beta z} + V^-(0)e^{+j\beta z}$$

# Voltages and Currents on a Transmission Line



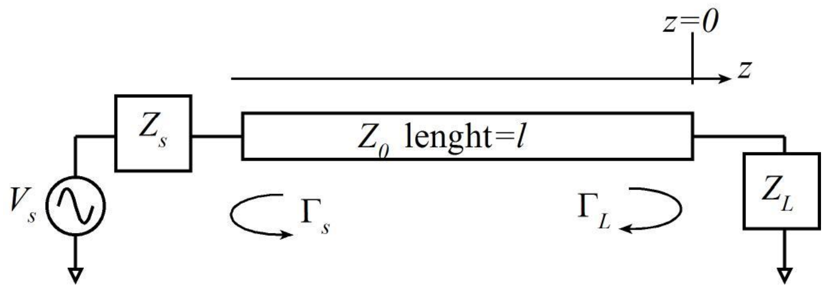

Phasor voltage on the line :

$$V(z) = V^+(z) + V^-(z) = V^+(0)e^{-j\beta z} + V^-(0)e^{+j\beta z}$$

Phasor current on the line :

$$Z_0 I(z) = V^+(z) - V^-(z) = V^+(0)e^{-j\beta z} - V^-(0)e^{+j\beta z}$$

# Wave Parameters




Define wave amplitude $a$ such that if $\| a \| = 1$, then wave power $= 1$ Watt.

Voltage in forward wave: $V^+(z)$

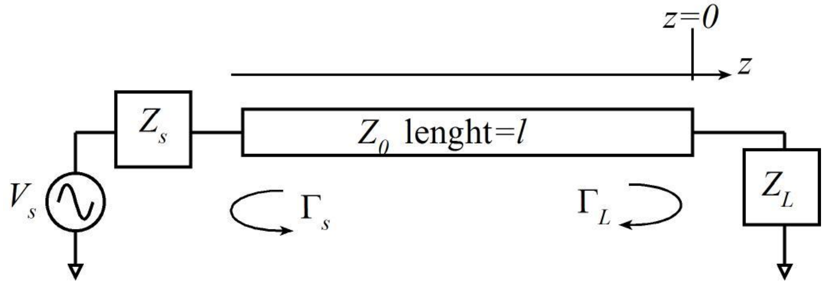
Current in forward wave: $I^+(z) = V^+(z) / Z_0$

Power in forward wave $= V^+(I^+)^* = \| V^+(z) \|^2 / Z_0$

Foward wave amplitude: $a(z) = V^+(z) / \sqrt{Z_0}$

Reverse wave amplitude: $b(z) = V^-(z) / \sqrt{Z_0}$

# Wave Parameters and Power

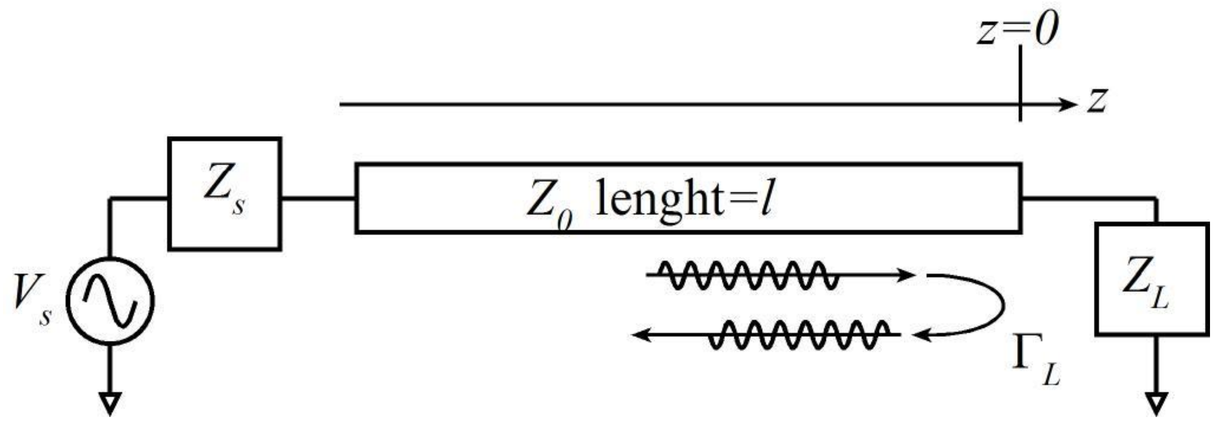



$$\text{Power in forward wave} = V^+(I^+)^* = \| V^+(z) \|^2 / Z_0 = a(z)a^*(z)$$

$$\text{Power in reverse wave} = V^-(I^-)^* = \| V^-(z) \|^2 / Z_0 = b(z)b^*(z)$$

Throughout the notes, we use R.M.S. quantities.

# Relfections from the Load

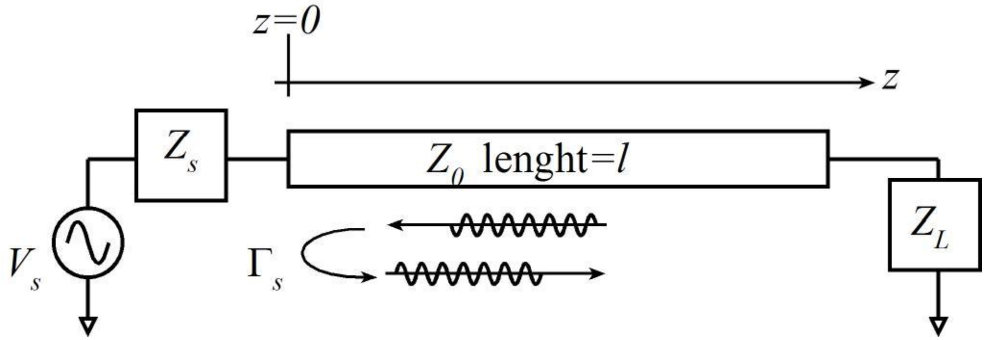



$$V^-(0) = \Gamma_L V^+(0)$$

where $\Gamma_L = \dfrac{z_L - 1}{z_L + 1}$ is the load reflection coefficient.

and $z_L = \dfrac{Z_L}{Z_0}$ is the *normalized* load impedance.

# Relfections from the Generator




$$V^+(0) = \Gamma_S V^-(0) + T_s V_s$$

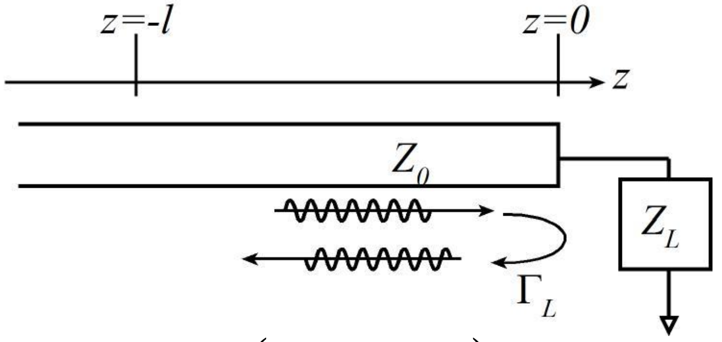
where $T_S = \dfrac{Z_0}{Z_0 + Z_S}$ is the source transmission coefficient

where $\Gamma_S = \dfrac{z_S - 1}{z_S + 1}$ is the source reflection coefficient.

and $z_S = \dfrac{Z_S}{Z_0}$ is the *normalized* source impedance.

Note that the reference plane $(z = 0)$ has been moved.

# Movement of Reference Plane




$$V(z) = V^+(z) + V^-(z) = V^+(z) \cdot \left(1 + \Gamma(z)\right)$$

where $\Gamma(z) \equiv \dfrac{V^-(z)}{V^+(z)}$ is the position-dependent reflection coefficient

$$V(z) = V^+(0)e^{-j\beta z} \cdot \left(1 + \Gamma(0)e^{+2j\beta z}\right)$$

because $\Gamma(z) \equiv \dfrac{V^-(z)}{V^+(z)} = \dfrac{V^-(0)e^{+j\beta z}}{V^+(0)e^{-j\beta z}} = \Gamma(0)e^{+2j\beta z}$

# Position-Dependent Reflection Coefficient

Reflection coefficient at a distance $l$ from load.

$$\Gamma(-l) = \Gamma(0)e^{+2j\beta z}$$

The reflection coefficient has gone through a phase shift of

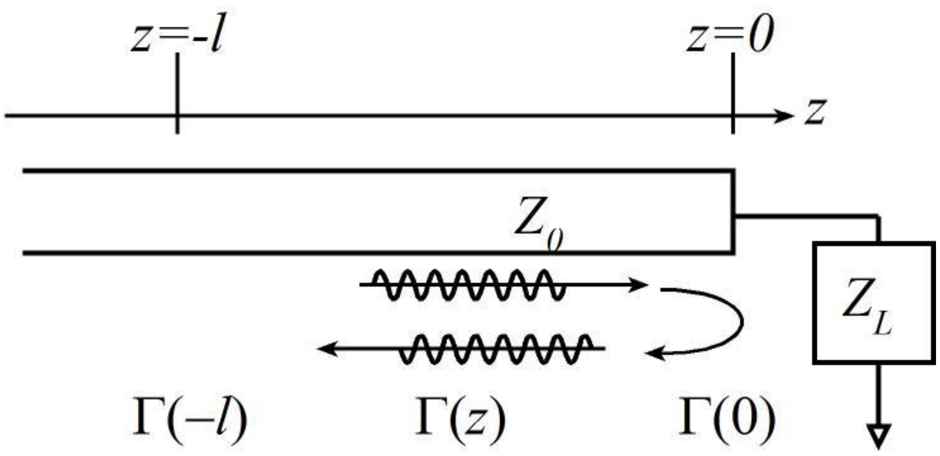
negative $\dfrac{l}{\lambda}\cdot 2\cdot 2\pi$ radians.

or

negative $2\cdot\beta\cdot l$ radians.

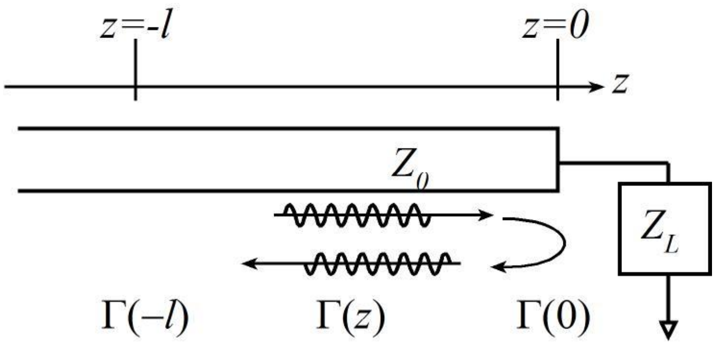
or

negative $\dfrac{l}{\lambda}\cdot 2\cdot 360$ degrees.




...simply because $V^{+}$ and $V^{-}$ undergo 360 degree phase shifts every wavelength of distance.

# Impedance vs. Position




Impedance at any point

$$Z(z) \equiv V(z)/I(z) = \left(V^+(z)+V^-(z)\right)/\left(I^+(z)-I^-(z)\right)$$

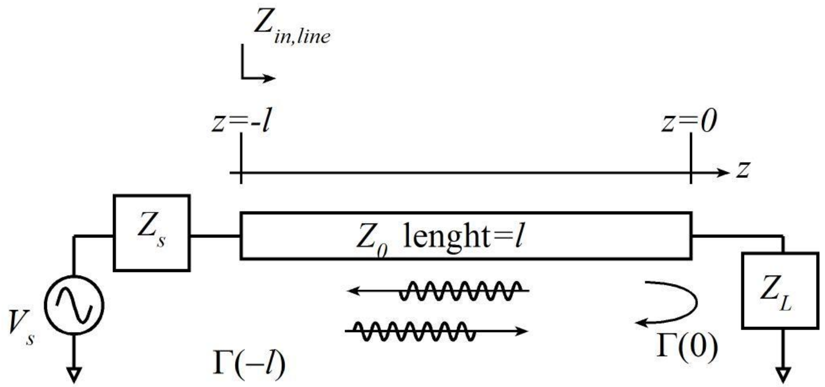
$$= Z_0 \cdot \left(V^+(z)+V^-(z)\right)/\left(V^+(z)-V^-(z)\right)$$

$$Z(z) = Z_0 \cdot \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

Normalized impedance at any point

$$\mathcal{Z}(z) \equiv Z(z)/Z_0 = \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

# Line Input Impedance

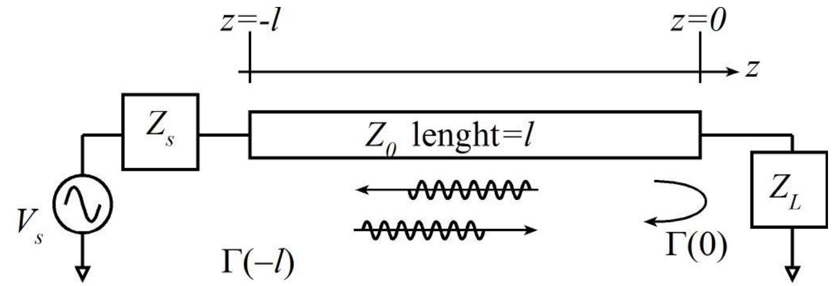



Input impedance at $z = -l$

$$\mathcal{Z}(-l) = \frac{V(-l)}{I(-l)} = \frac{1+\Gamma(-l)}{1-\Gamma(-l)} \text{ normalized.}$$

$$Z(-l) = \frac{V(-l)}{I(-l)} = Z_0 \cdot \frac{1+\Gamma(-l)}{1-\Gamma(-l)} \text{ unnormalized.}$$

# Impedance and Reflection Coefficient vs Postion



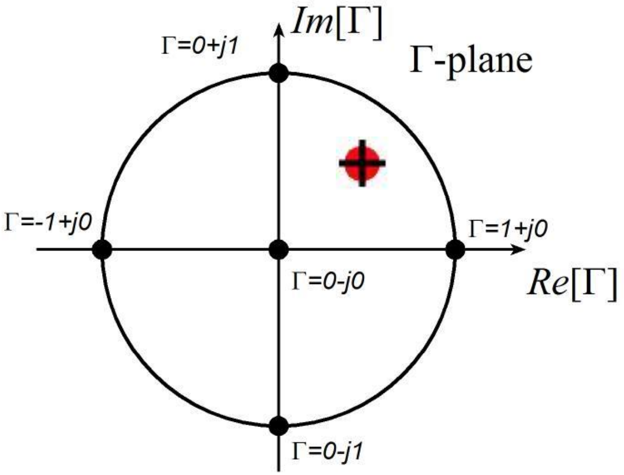

$$\left.\begin{aligned} V(z) &= V^+(z) + V^-(z) = V^+(0)e^{-j\beta z} + V^-(z)e^{+j\beta z} \\ Z_0 I(z) &= V^+(z) - V^-(z) = V^+(0)e^{-j\beta z} - V^-(z)e^{+j\beta z} \end{aligned}\right\} \quad \text{waves}$$

$$\left.\begin{aligned} \Gamma(z) &= V^+(z)/V^-(z) \\ \Gamma(z) &= \Gamma(0)\, e^{+2j\beta z} \end{aligned}\right\} \quad \text{reflection coefficients}$$

$$\left.\mathfrak{z}(z) = \frac{1}{Z_0}\frac{V(z)}{I(z)} = \frac{1+\Gamma(z)}{1-\Gamma(z)}\right\} \quad \text{normalized impedance}$$

Conceptually simple, but tedious math. $\rightarrow$ Work with a graphical tool.

# Developing the Smith Chart

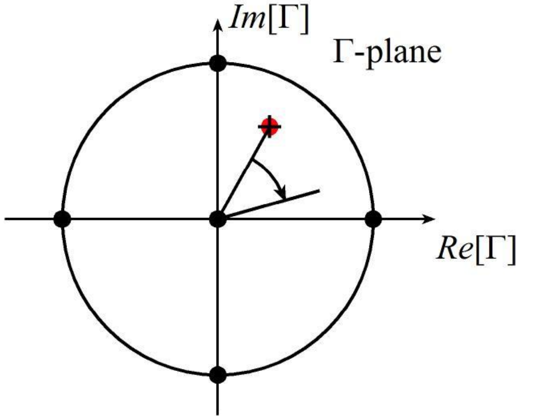



The relationship $z = \dfrac{1+\Gamma}{1-\Gamma} \leftrightarrow \Gamma = \dfrac{z-1}{z+1}$ is key.

The relationship is a 1-1 mapping between the complex #s $z$ and $\Gamma$; a conformal transformation. This relationship can be graphed.

In the 2-dimensional plane of $\Gamma$ - the $\Gamma$ plane - a reflection coefficient is represented by a point. (here, a red dot).

# Moving Refernce Planes---on the Smith Chart



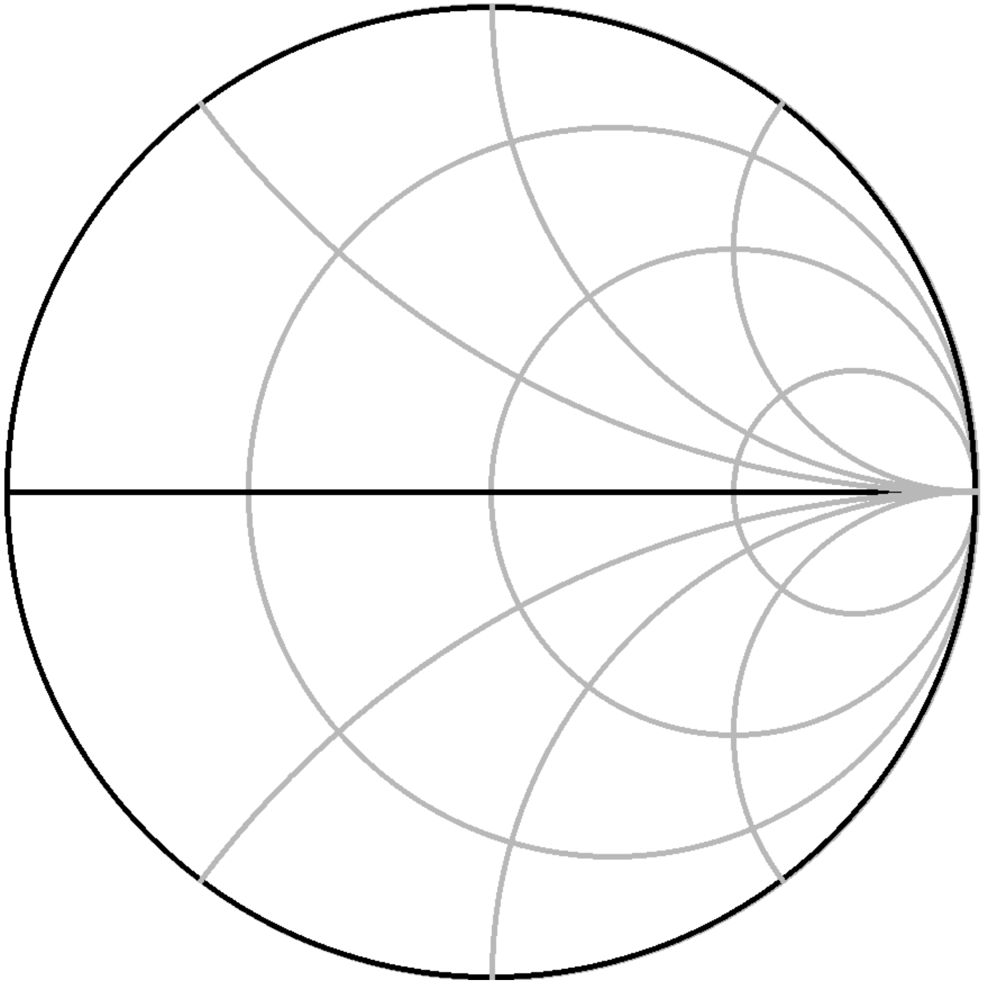

As we move a distance $l$ away from the load, the vector $\Gamma$ rotates by an angle $\Delta\theta$

$$\Delta\theta = -2\beta l$$

$$= -360^{\circ} \cdot \frac{l}{\lambda} \cdot 2$$

= one whole rotation in the $\Gamma$ plane for each half-wavelength movement on the transmission line.

# Finding Impedances



$$\mathfrak{z}(z) = \frac{1}{Z_0}\frac{V(z)}{I(z)} = \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

This is a 1:1 relationship

between reflection coefficient $\Gamma$ (magnitude and phase)

and normalized impedance $\mathfrak{z}$ (real and imaginary parts).

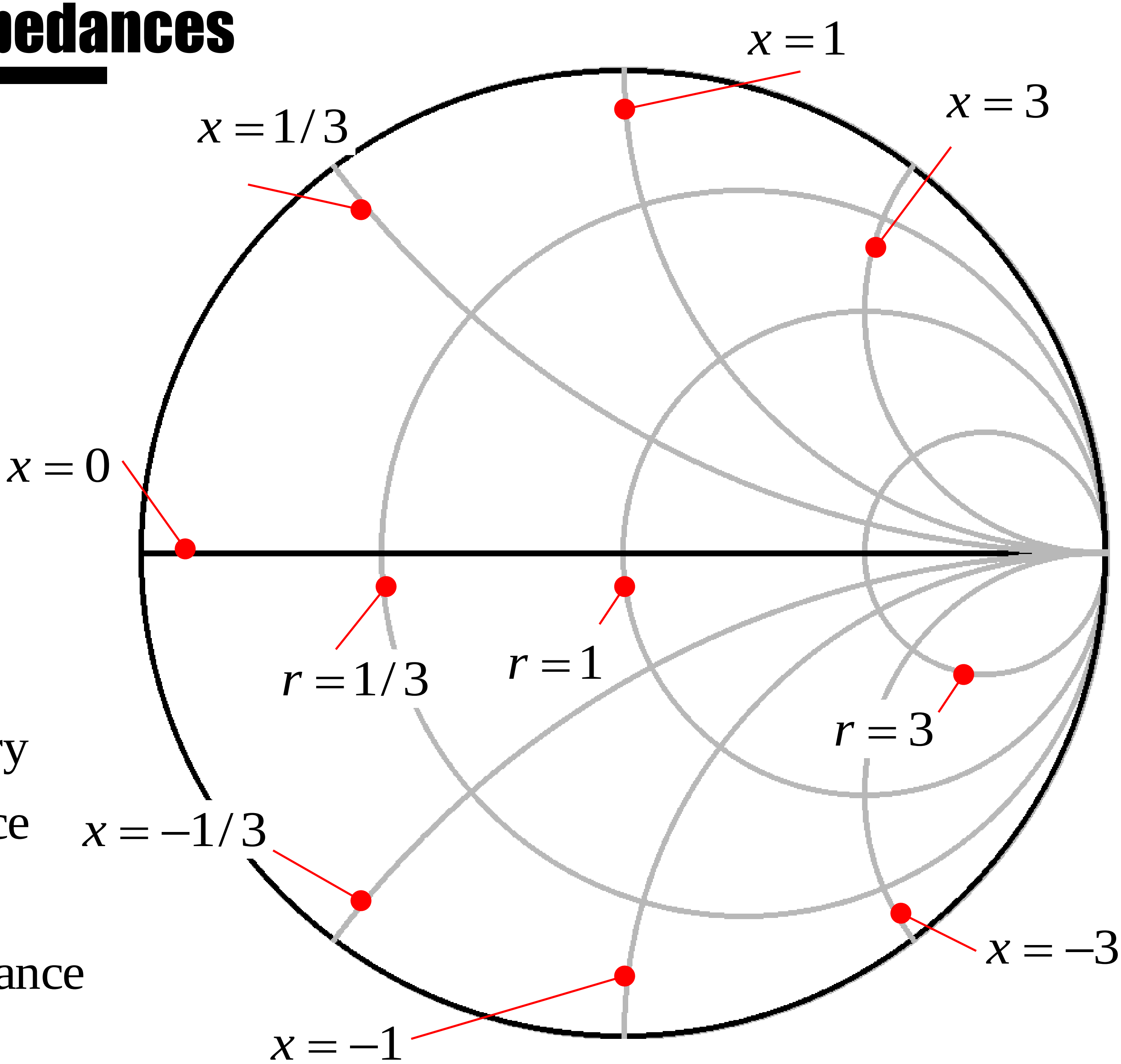
Plot the units of $\mathfrak{z}$ on the $\Gamma$ plane!

# Finding Impedances

$$\mathfrak{z} = \frac{1+\Gamma}{1-\Gamma} = \frac{Z}{Z_0}$$

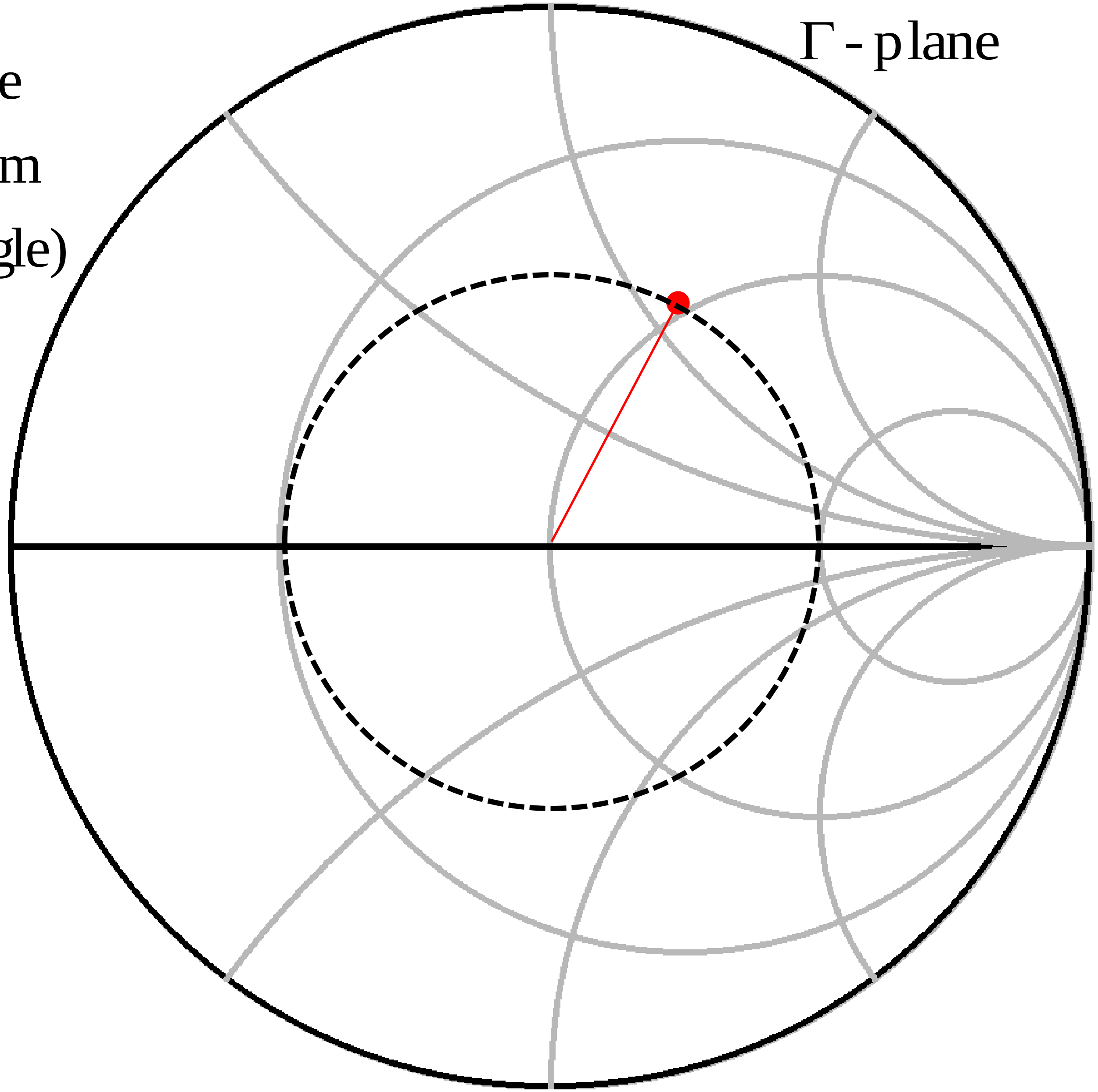
$$Z = R + jX$$

$$\mathfrak{z} = r + jx$$




Real and imaginary parts of impedance can be read from the curved impedance axes on the chart.

# Finding Reflection Coefficient

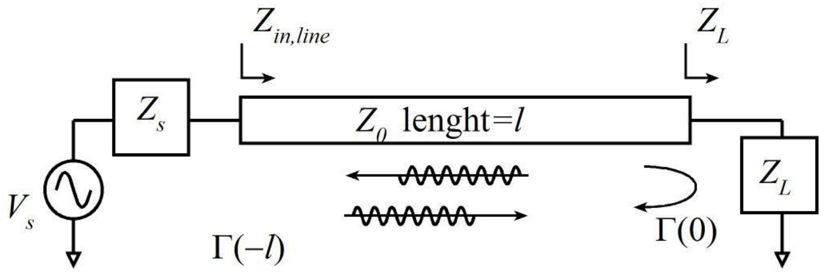



The magnitude and angle of Γ are simply read from the chart (radius and angle)

this measurement can be done using a ruler and a protractor*.

* though today the CAD softaware does the measurement from a cursor.

# Using the Smith Chart

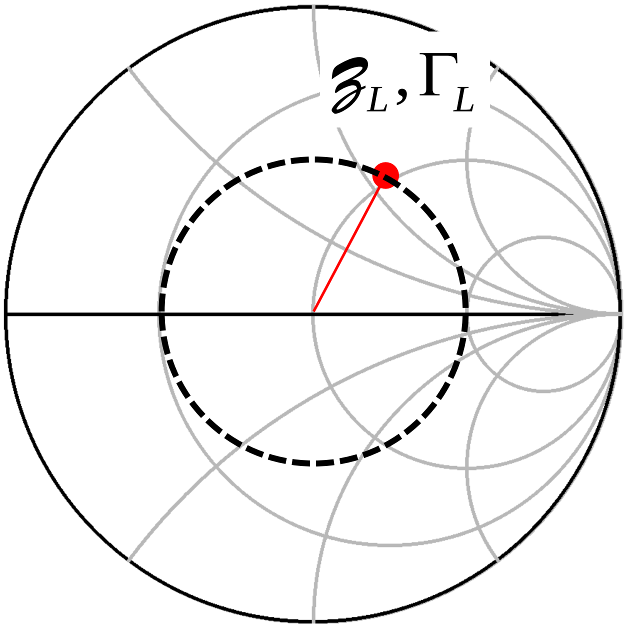



Starting with the load impedance $Z_L$,

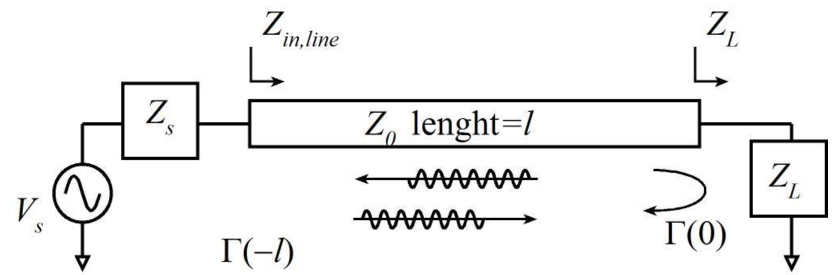
we compute $z_L = Z_L / Z_0$.

We then find this point on the Smith chart.

This determines the load reflection coefficient $\Gamma_L$.



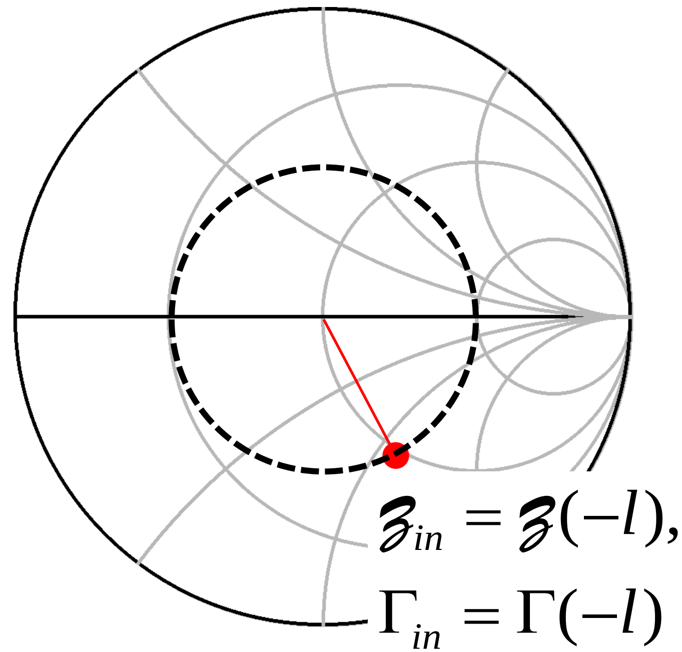
# Using the Smith Chart



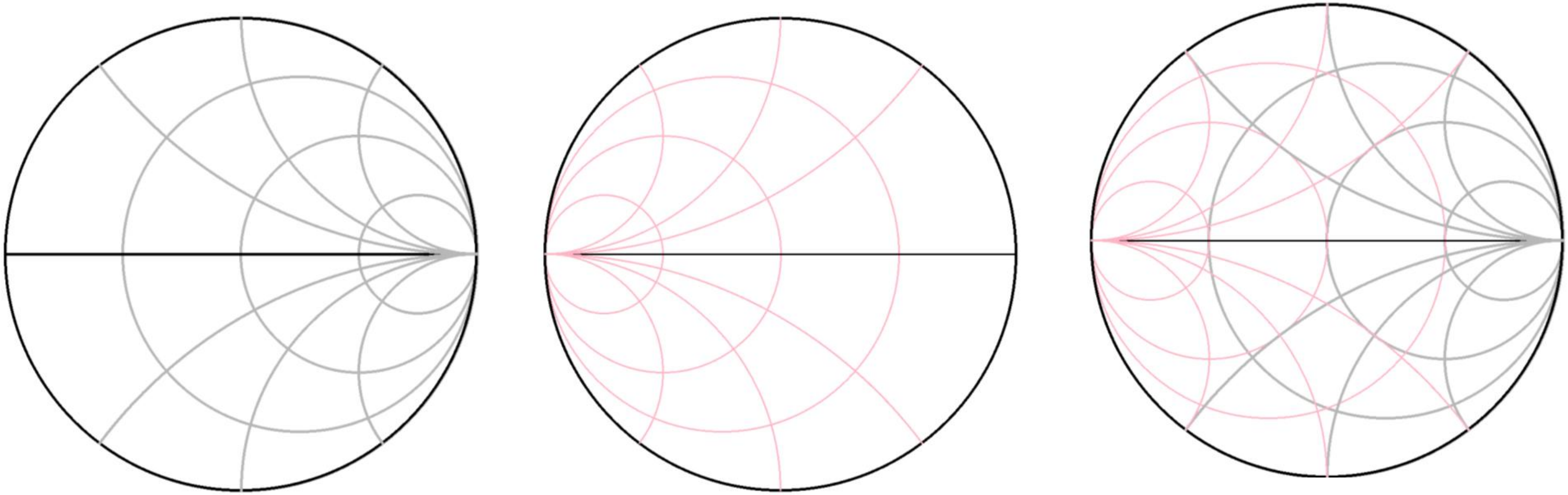

We then rotate the vector $\Gamma$ through an angle $360^o \cdot (2l/\lambda)$.

This locates the input reflection coefficient.

We can now read off the input impedance.



# Impedance-Admittance Chart



Impedance $Z = R + jX$

Normalized impedance $z = Z / Z_o = r + jx$

Admittance $Y = 1/Z = G + jB$

Normalized impedance $y = YZ_o = Y / Y_o = g + jb$

Smith charts can have axes for $z$, $y$, or both.

# Solving Wave Equations Quickly

# Waves and Fourier Transforms (1)

Maxwell's equations give us a wave equation :

$$\nabla^2 \vec{E} = \mu\varepsilon \cdot \frac{\partial^2 \vec{E}}{\partial t^2} + \mu \frac{\partial \vec{J}}{\partial t} + \varepsilon^{-1} \vec{\nabla} \rho \text{ if } \mu \text{ and } \varepsilon \text{ are uniform}$$

Assume nonzero conductivity, $\vec{J} = \sigma \vec{E}$, assume charge neutrality $\rho = 0$.

$$\nabla^2 \vec{E} = \mu\varepsilon \cdot \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

To solve this easily, assume

$$E_x(x, y, z, t) = E_x e^{j\omega t} e^{-jk_x x} e^{-jk_y y} e^{-jk_z z} \text{ (and the same for } E_y, E_z)$$

Sometimes, the $k$'s are complex : writing $\gamma = jk = \alpha + j\beta$

$$E_x(x, y, z, t) = E_x e^{j\omega t} e^{-\gamma_x x} e^{-\gamma_y y} e^{-\gamma_z z}$$

# Waves and Fourier Transforms (2)

So we have:

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}=\mu\varepsilon\cdot\frac{\partial^2 E_x}{\partial t^2}+\mu\sigma\frac{\partial E_x}{\partial t} \text{ (and the same for } E_y, E_z\text{)}$$

Given

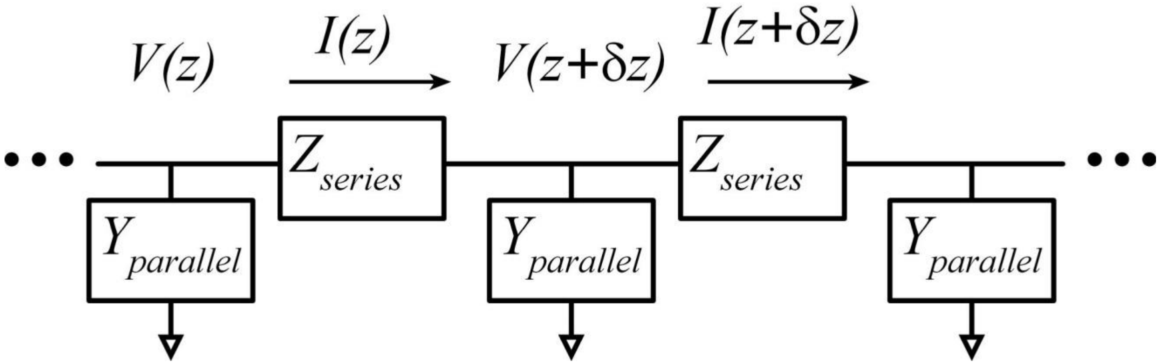
$$E_x(x,y,z,t)=E_x e^{j\omega t}e^{-jk_x x}e^{-jk_y y}e^{-jk_z z} \text{ (and the same for } E_y, E_z\text{)},$$

This becomes simply

$$k_x^2+k_y^2+k_z^2=k^2 \text{ where } k^2=(j\omega\mu)(j\omega\varepsilon+\sigma)$$

This is the wave equation in the sinusiodal steady state

# Waves and Fourier Transforms (3)




Now consider a 1-dimensional system (transmission-line)

$$\frac{\partial V}{\partial z} = -Z_{series} I \quad \text{and} \quad \frac{\partial I}{\partial z} = -Y_{parallel} V$$

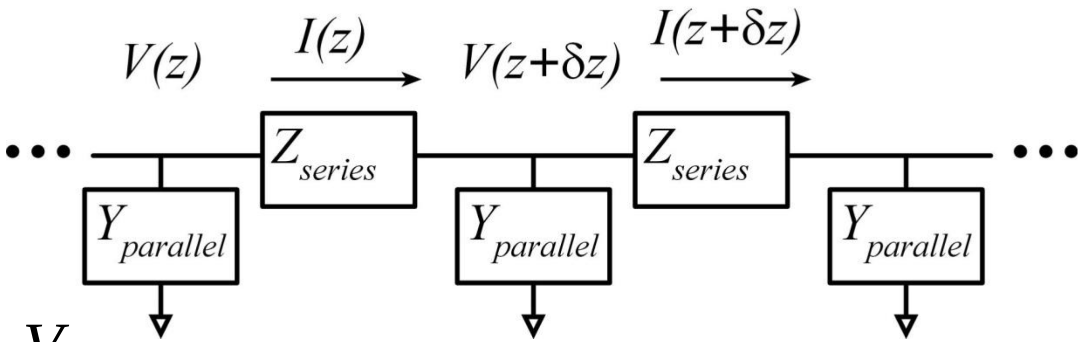
To solve this easily, assume

$$V^{+/-}(z,t) = V^{+/-} e^{j\omega t} e^{-\gamma z} \quad \text{and} \quad I^{+/-}(z,t) = I^{+/-} e^{j\omega t} e^{\gamma z}$$

Then

$$\gamma V = Z_{series} I \quad \text{and} \quad \gamma I = Y_{parallel} V$$

# Waves and Fourier Transforms (4)




$$\gamma V = Z_{series} I \quad \text{and} \quad \gamma I = Y_{parallel} V$$

Multiply these:

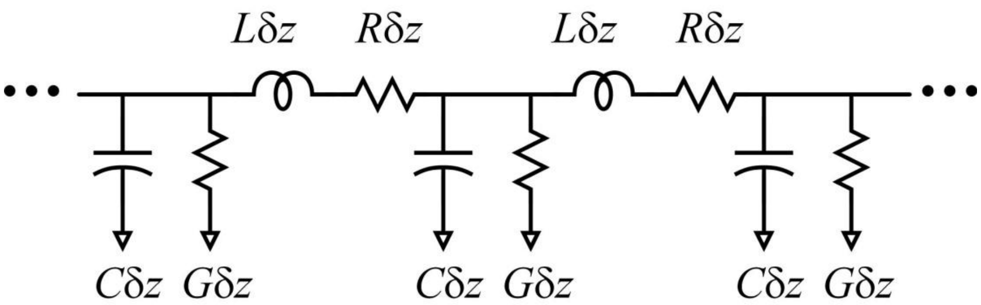
$$\gamma^2 VI = Z_{series} Y_{parallel} VI \rightarrow \gamma = \pm\sqrt{Z_{series} Y_{parallel}}$$

Divide these:

$$V^{\pm} / I^{\pm} = Z_{series} I^{\pm} / Y_{parallel} V^{\pm} \rightarrow Z_0 \equiv (V^{\pm} / I^{\pm}) = \pm\sqrt{Z_{series} / Y_{parallel}}$$

The $\pm$ before the root indicates that the forward current has the same sign as the forward voltage, while the reverse current has sign opposite that of the reverse voltage.

# Waves and Fourier Transforms (5)




Line has series inductance $L$ and series $R$ resistrance per unit lenght.

Line has parallel capacitance $C$ and parallel conductance $G$ per unit lenght.

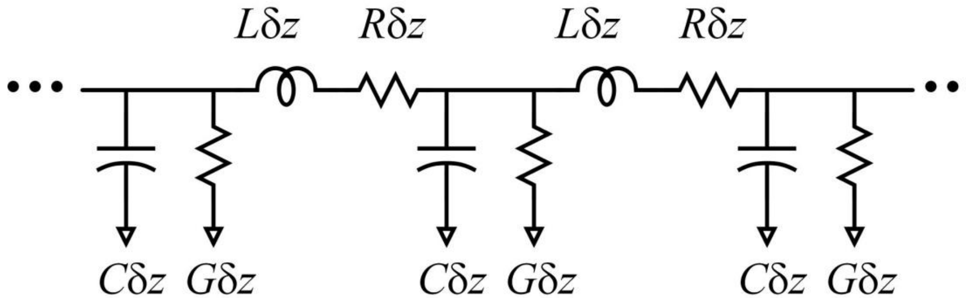
Then

$$Z_{series} = R + j\omega L \quad Y_{parallel} = G + j\omega C$$

So :

$$\gamma = \pm\sqrt{(R + j\omega L)(G + j\omega C)} \quad Z_0 = \pm\sqrt{(R + j\omega L)/(G + j\omega C)}$$

# Waves and Fourier Transforms (6)




$$\gamma = \pm\sqrt{(R + j\omega L)(G + j\omega C)} \qquad Z_0 = \pm\sqrt{(R + j\omega L)/(G + j\omega C)}$$

Suppose $R << \omega L$ and $G << j\omega C$. Use $(1+\varepsilon)^N = 1 + N\varepsilon + O(\varepsilon^2)$

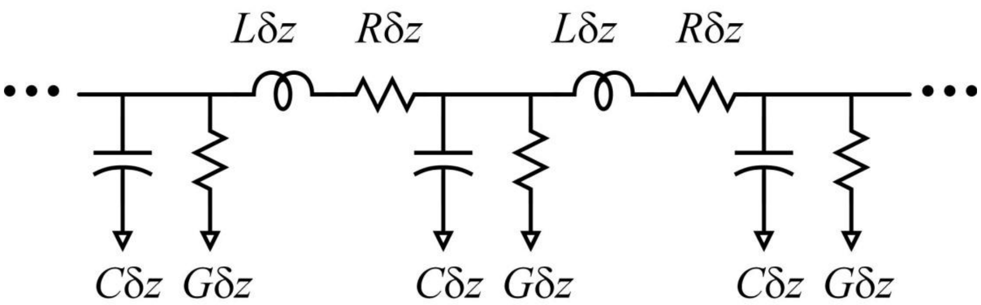
$$Z_0 = \pm\sqrt{j\omega L / j\omega C}\sqrt{(1 + R / j\omega L)/(1 + G / j\omega C)}$$

$$Z_0 \cong \pm\left[\sqrt{\frac{L}{C}} \cdot \frac{1 + R / j2\omega L}{1 + G / j2\omega C}\right]$$

Note that $Z_0$ becomes slightly complex.

Important sometimes in S-parameter calibration

# Waves and Fourier Transforms (7)




$$\gamma = \pm\sqrt{(R + j\omega L)(G + j\omega C)}$$

Suppose $R << \omega L$ and $G << j\omega C$. Use $(1+\varepsilon)^N = 1 + N\varepsilon + O(\varepsilon^2)$

$$\gamma = \pm\sqrt{(j\omega L)(j\omega C)}\sqrt{1 + R / j\omega L}\sqrt{1 + G / j\omega C}$$

$$\cong \pm j\omega\sqrt{LC} \cdot (1 + R / j2\omega L)(1 + G / j2\omega C)$$

$$\gamma \cong \pm\left[\frac{R}{2\sqrt{L / C}} + \frac{G\sqrt{L / C}}{2} + j\omega\sqrt{LC}\right]$$

$$\gamma \cong \pm\left[\frac{R}{2Z_0} + \frac{GZ_0}{2} + j\omega\sqrt{LC}\right] = \pm[\alpha + j\beta]$$