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Chapter 1

THE CHARACTERISTICS OF CIRCUIT ELEMENTS AT HIGH FREQUENCIES—M.McWhorter

SECTION 1–1 LUMPED ELEMENTS

The purpose of this section is to acquaint the student with the behavior of "ordinary" circuit elements at the higher frequencies. A good place to begin is to define what is meant by high frequencies in the context of this course. To a mechanical engineer high frequencies might be a relatively few Hertz while to a microwave engineer high frequency might mean 10–30 GHz so some sort of definition for our purposes is needed. We are going to consider the frequency range where the "pure" circuit elements (R,L, and C) are hard to realize because of parasitic elements or where elements are best thought of in terms of distributed parameters like transmission lines or line resonators. However, we shall not consider elements where modes other than the TEM (Transverse Electromagnetic Mode) are important. This range of frequencies includes frequencies where transmission line theory is useful, but we shall not consider waveguides or cavity resonators. The actual frequency limits depend upon the size of structures that can be built: for example, using hybrid circuits and chip transistors it is quite possible to consider frequencies of 3–5 GHz or more. Because of our test equipment we shall limit our work to <1300 MHz[1].

1–1.1 RESISTORS AT HIGH FREQUENCIES

A resistor made of a slab of resistive (or semi–conducting) material has the d–c resistance shown in Fig. 1–1. The ends are assumed to be equi–potentials and in practice might be a conducting material fired or painted onto the resistive material. The d–c resistance of the resistor is given by the equation in Fig. 1–1. Such a slab resistor is similar to those actually used in hybrid circuits, and it or the cylindrical version shown in Fig. 1–2 are useful in visualizing the problems that are likely to exist at higher frequencies. Note that the current flowing through the resistor will set up magnetic lines of force around the resistor so that energy will be stored in the external magnetic

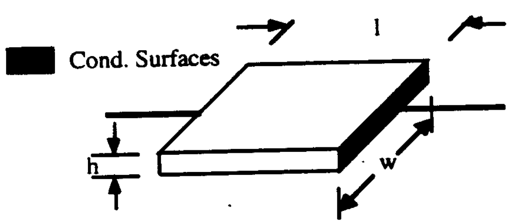

Fig. 1–1: Geometry of a slab resistor. The resistance R = r•l/w•h where r = the resistivity of the material in Ω–cm.

field. This effect is normally accounted for by assuming a series inductance which accounts for the energy storage of a non–ideal resistor. In addition the current flowing through the resistor produces a voltage drop which in turn sets up an electrostatic field across the resistor. This field also stores energy, this time in proportion to the square of the voltage across the resistor. Such energy storage may be accounted for by considering the ideal resistor to have a parallel capacitor. Both the capacitor and the inductor are functions of the physical geometry of the resistor; e.g., making the resistor shorter reduces the inductance (see under the discussion of inductors.)

The equivalent circuit for the resistor thus formed is shown in Fig. 1–3. The effects of the

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[1]For an explanation of the TEM mode see Ref. 1, pp. 41-44. This reference also shows the effect of higher order modes.

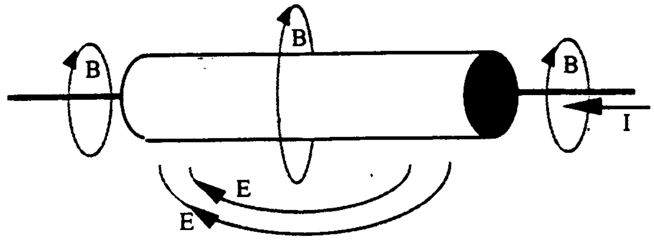


Fig. 1–2: Fields about an ordinary cylindrical resistor [like a carbon composition resistor.]

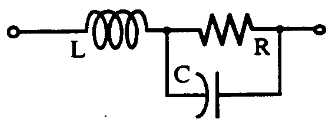


Fig. 1–3: An approximate equivalent circuit for a resistor including parasitic elements.

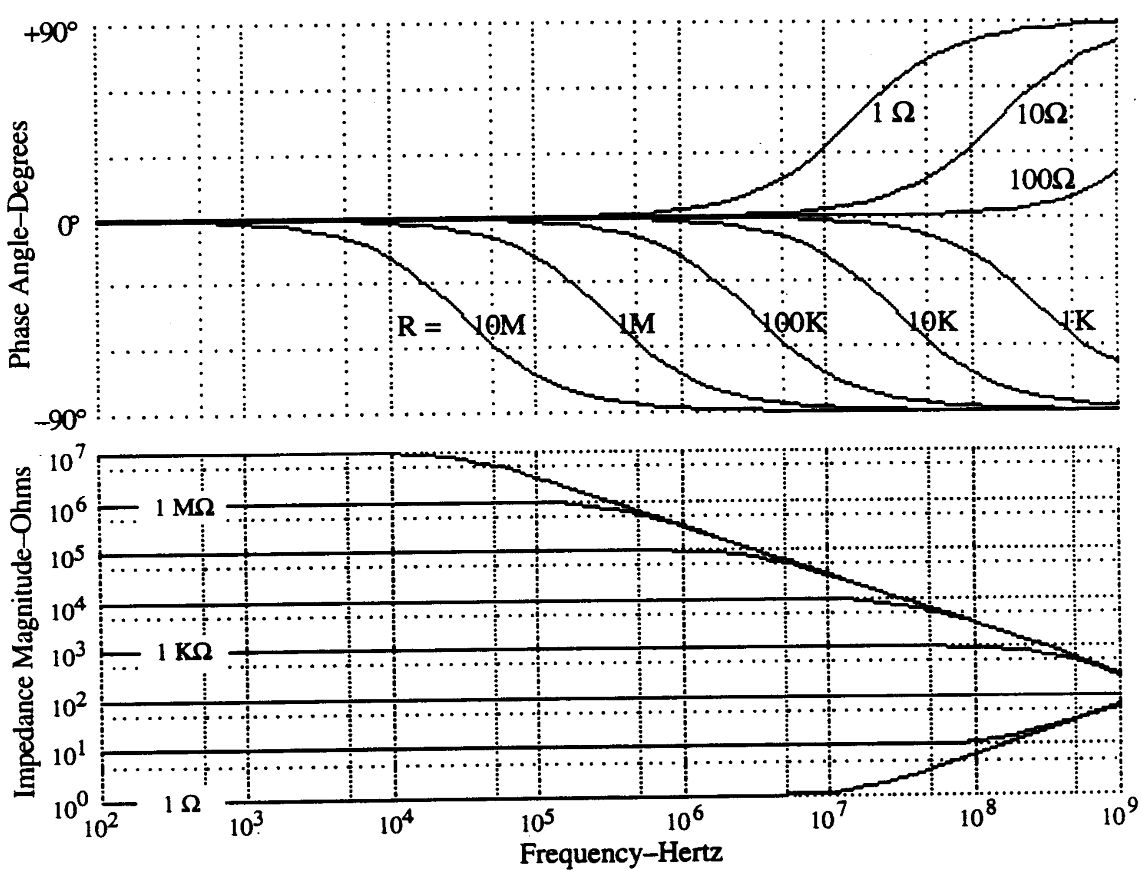


Fig. 1–4; The impedance and phase of various values of a resistor vs frequency. The assumed shunt capacitance is 0.5 pF and the series inductance is 10 nH.

parasitic elements L and C depend upon their values and the value of the resistor. A typical 1/4 W resistor has about 10 nH of inductance (if the total length including the leads is about 1 cm) and 0.5 pF of capacitance[1]. Figure 1–4 shows the magnitude and phase of the impedance versus frequency for various values of R. Ideally Z should not vary with frequency. Note that for high values of R the shunt capacitance predominates—for the case of the 1 MΩ resistor the impedance is becoming capacitive below 100 KHz. (The magnitudes of the impedance shown assume that you don't worsen the situation by adding more capacitance or inductance from your wiring.)

A low resistance like 10 ohms is not affected by the shunting capacitance until f > 1 GHz, but the series inductance does increase the effective impedance at high frequencies. Note that for resistors in the range of 100 ohms the inductive and capacitive effects tend to balance and the impedance value stays relatively constant over a wide range of frequency. This is one reason wideband circuits tend to be built around this impedance level. Another reason is the availability of coaxial lines in this impedance range.) At very high frequencies the series inductance predominates using this particular equivalent circuit; however, at these frequencies the circuit is inadequate because the elements are in actuality distributed.

The preceding analysis has two important defects that must be recognized: the first, as mentioned, is that both L and C are really distributed elements, and the lumped circuit is an idealization. The second is that at higher frequencies the current does not flow uniformly throughout the body of the resistor. This manifestation of skin effect tends to increase the resistance with frequency, and will be discussed further in connection with transmission lines.

The value of C can be measured with a bridge capable of measuring parallel components at high frequencies. The value of R can be measured using a r–f bridge capable of measuring series components or using the S–Parameter test set.

1–1.2 CAPACITORS AT HIGH FREQUENCIES

A capacitor may be defined in terms of the ratio of charge stored (Q) to voltage across the capacitor (V) by C = Q/V. [If the capacitor is non–linear, it may be defined at some operating point V_0 as C = dQ/dV.] An ideal capacitor stores energy in an electric field only and with no loss. The energy stored is $W = 0.5 \cdot C \cdot V^2$. At low frequencies the loss is often expressed as a power factor. If the loss is thought of as a series resistor (r_s) then the power factor is given by:

$$PF = \cos[\tan^{-1}(2 \cdot \pi \cdot f \cdot r_s \cdot C)] = \cos[\tan^{-1}(\omega \cdot r_s \cdot C)] \quad [1\text{–}1]$$

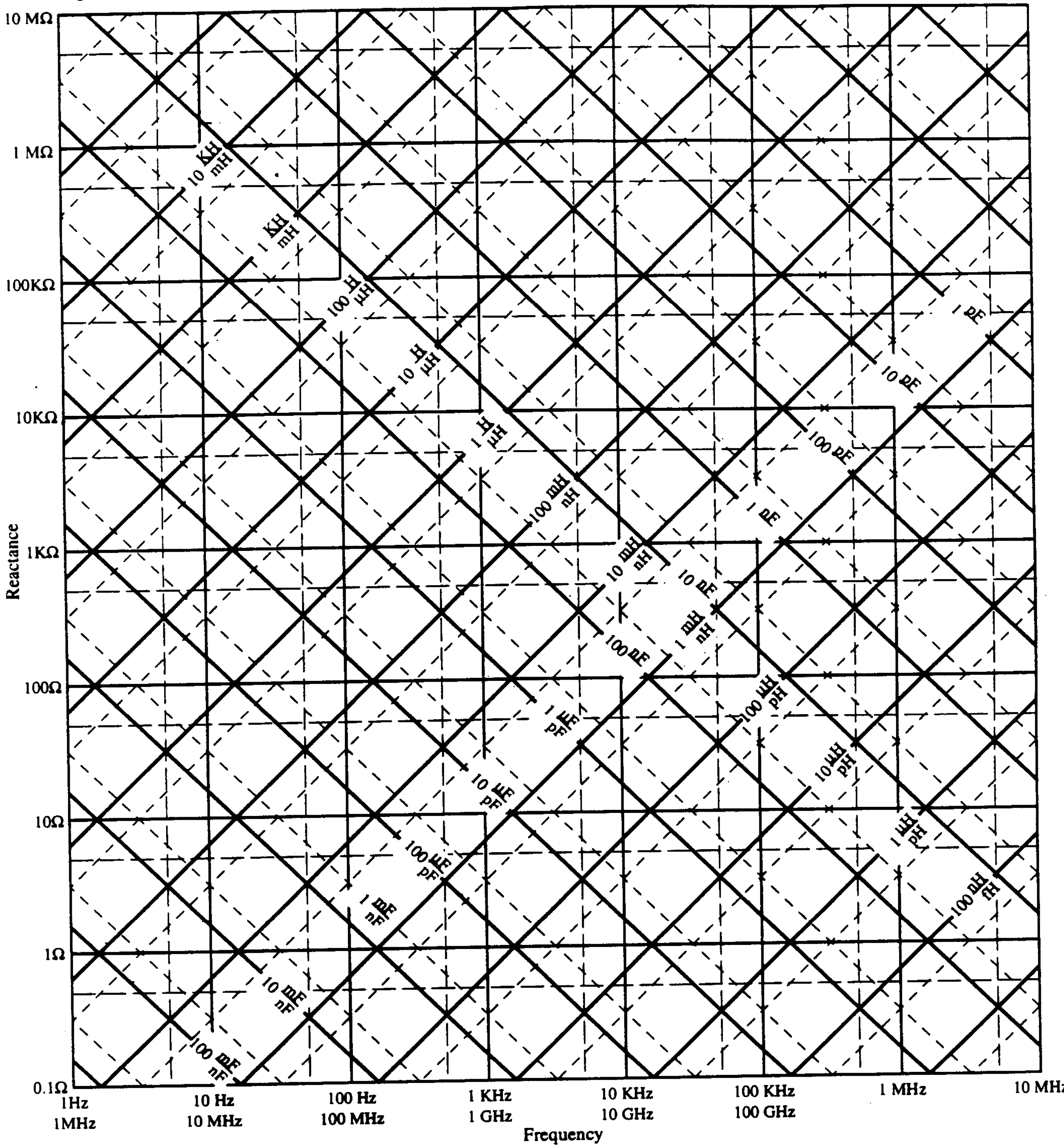
$$\cong \omega \cdot r_s \cdot C \quad [1\text{–}2]$$

For lowloss capacitors where the power factor (≅ the dissipation factor, also) is almost exactly 1/Q where Q is the capacitor Q. Capacitor losses may also be represented by a parallel resistor, but this is usually done only at low frequencies. [Note that the value of the resistor for the two representations is very different.]

The loss in a capacitor arises from the resistance in the metallic plates and leads, and from the dielectric loss of the material between the plates. The dielectric loss of a wide range of materials is given in Ref. 8. Some representative values of the dissipation factor (D) for dielectrics widely used for insulators and capacitors are given in Table 1–1 where D is measured at 100 MHz and K

[1]Typical values of the parasitics for a small chip resistor are L = 1.2 nH, C = 0.03 pF. These values were measured at Hewlett Packard [HP]. Since this type of element is largely used with strip lines, the resistor may also be treated as a lossy transmission line.

Fig. 1–5: Reactance Chart



The chart may be used to find the reactance of a given inductor or capacitor at a given frequency, or the resonant frequency of a given inductance and capacitance.

To Find Reactance: Enter the charts vertically from the bottom (frequency) and along the lines slanting upward to the left (inductance) or to the right (capacitance). Corresponding scales (upper or lower) must be used throughout. Project horizontally to the left from the intersection and read reactance.

To Find Resonant Frequency: Enter the slanting lines for the given inductance and capacity. Project downward from their intersection and read the resonant frequency from the bottom scale. Corresponding scales (upper or lower) must be used throughout.

From the *Radio Engineers Handbook*, F. E. Terman

Table 1-1

Mat'l	K	D-%
Air	1.00	0.00
Epoxy-glass	4.6—5.9	3.0
Formica XX[1]	4.04	5.7
Polyethylene	2.26	0.02
Polystyrene	2.55	<0.01
Quartz	3.75	0.02

is the dielectric constant. The capacitance of two parallel plates separated by a dielectric is given by

$$C = 0.0884 \cdot K \cdot A/d \qquad \text{[C in pF]} \qquad [1\text{–}3]$$

Where K = dielectric constant from Table 1–1, A = area of plates in cm^2, and d = distance between plates in cm. The capacitance is in pF. This equation neglects the fringing fields and therefore gives a somewhat low value for C.

A 1 cm square section of copper on a 1/16" (0.159 cm) thick epoxy glass board with the average dielectric constant of 5.0 gives 2.8 pF. At 100 MHz this represents only 570 ohms of capacitive reactance. [A convenient way to estimate reactances and resonant frequencies is to use the reactance chart of Fig. 1–5[2].] Equations for the calculations of the capacitance of many other electrode configurations are given in F.E. Terman, Ref. 7, pp. 109–119.

In addition to the loss of a non-ideal capacitor, there will be inductance due to the leads and perhaps the electrodes themselves if the capacitor is poorly made for h-f use. Therefore the equivalent circuit for a non-ideal capacitor looks like Fig. 1–6 where the loss resistor (R), the lead inductance (L), and the desired capacitor (C) are all in series. The effect of L is to increase the apparent capacitance (i.e., lower the effective impedance) for frequencies less than the series resonant frequency, f_0:

$$f_0 = \frac{1}{2 \cdot \pi \cdot \sqrt{L \cdot C}} \qquad [1\text{–}4]$$

At f_0, $Z = R$, and above f_0 the capacitor acts like an inductor approximately equal to the inductance of a wire connecting the two terminals of the capacitor (assuming little <u>internal</u> inductance in the capacitor which is usually true in a good h-f capacitor.) This reversal of the behavior of the capacitor at high frequencies can be very troublesome and is well illustrated in Fig. 1–7. The ideal capacitor shows a linear decrease in impedance with frequency (on log-log scales). The non-ideal capacitor in this example is: C = 10 nF, R = 0.5 Ω, and L = 10 nH—values typical for a small ceramic disc capacitor with 1 cm total lead length[3]. The resonant frequency of this combination is 15.9 MHz, and the impedance is a minimum at this frequency. Above 15.9 MHz this capacitor and

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[1] Formica XX is also known as a phenolic-paper dielectric which is commonly used for cheap circuit boards. Its high loss makes it unsuitable at high frequencies.

[2] The chart may be used to find the reactance of both inductors and capacitors as well as the resonant frequency of their combination. The directions for use of the chart on on the bottom of the figure.

[3] Typical values of the parasitic elements for a NPO ceramic chip capacitor are 2 nH and 0.2 Ω [HP communication]. For a 100 pF capacitor this results in a self resonant frequency of 360 MHz.

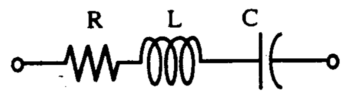

Fig. 1–6: An equivalent circuit for a capacitor.

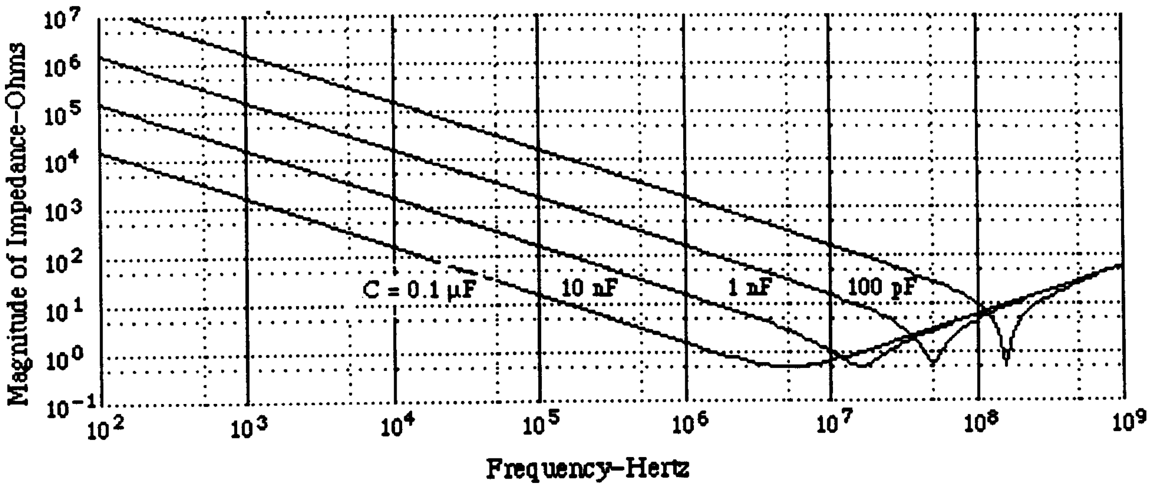

Fig. 1–7: The impedance of four different non–ideal capacitors. The assumed series resistance and inductance are the same for both: R = 0.5 Ω and L = 10 nH.

its leads behave like a 10 nH inductor. Note that using a larger capacitor [0.1 μF] decreases the impedance at low frequencies but not at high, assuming the same lead lengths. Sometimes the resonance can be used to advantage to obtain a very low impedance (as a bypass capacitor, for example) at one frequency. The performance at other frequencies may be worse, however.

In addition to these effects on the capacitor one should also know parameters such as the temperature coefficient, etc. These considerations are discussed in Ref. 8, but in general the best capacitors for stable h–f use are made with air [or vacuum], mica, low temp. coefficient. ceramic (NPO), or polystyrene dielectrics. Bypass capacitors which need only present a low impedance over wide ranges of frequency can be made of high–K ceramics which are usually not very temperature stable. The high–K ceramic capacitors have the advantage of being physically small for a given C; therefore their parasitic inductance can be kept small.

1–1.3 INDUCTORS AT HIGH FREQUENCIES

Inductance is that property of an electric circuit that opposes the change of current through the circuit. An inductor stores energy in a magnetic field. The energy stored is $W = L \cdot I^2/2$ where W is the stored energy in Watt–seconds or Joules. Any current carrying conductor has a magnetic field around it and therefore has self–inductance. The conductors may be geometrically arranged to increase the L over that of straight wire. The self–inductance of a straight wire (which must be a part of a closed path so that current can flow) at high frequencies where the current flows primarily on the surface of the wire is:

$$L = 0.00508 \cdot \text{L} \cdot [\text{Ln}(4 \cdot \text{L}/d) - 1] \qquad [\text{L in } \mu\text{H}] \qquad [1\text{–}5]$$

where L is in μH, L is the length of the wire in inches, and d is the wire diameter in inches. Some typical values for small lead wires are given in Table 1–2. (The 1 mil = 0.001 inch = 0.025 mm

Table 1–2

Wire Diam. (mils)	32	10	1.0
Wire gauge	#20	#30	—
Length, L			
0.5"	L = 8 nH	11 nH	17 nH
1.0"	L = 20 nH	25 nH	37 nH

diameter wire is typical for the internal lead into an integrated circuit.) In general, the inductance of a lead wire is minimized by keeping the wire as short and broad as possible.

To make a high quality air core inductor a common form is a cylinder wound with wire in a uniform spiral. If the radius of the wire spiral (measured to the center of the wire) is (r), the winding length (L), and the number of turns (n), the approximate inductance is

$$L = \frac{r^2 \cdot n^2}{9 \cdot r + 10 \cdot L} \qquad \text{[L in } \mu\text{H]} \qquad [1\text{–}6]$$

where L is in μH and the dimensions are in <u>inches</u>. The formula is accurate to 1% if $L > 0.8 \cdot r$, i.e., the coil is not too short. The inductance of many other configurations is given in Terman, Ref. 7.

A real inductor typically is less ideal than a good capacitor. The resistance of the wire produces loss which varies with frequency because of skin effect. The effect of the loss can be approximated by adding a series or shunt resistor to the equivalent circuit for the inductor. The quality factor or Q or the inductor is given by

$$Q = \frac{2 \cdot \pi \cdot f \cdot L}{R_s} = \frac{\omega \cdot L}{R_s} \qquad [1\text{–}7]$$

where R_s is the effective resistance *in series* with the inductor[1]. High Q is often desirable and is maximized by making the inductor large physically and using large diameter wire. Spacing the turns on the order of the wire diameter tends to increase the Q (and lowers the distributed capacity.) The Q of an inductor is very much a a function of frequency because of the ω in Eq. 1–7 and because R_s is proportional the square root of frequency due to skin effect. These two effects tend to make inductor Q rise with frequency which is true up to some frequency where other effects such as losses due to radiation cause the Q to decrease with frequency. There is, therefore, some frequency where the Q is a maximum for any inductor.

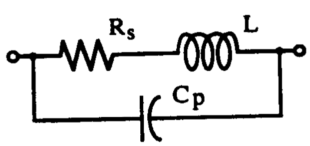

Fig. 1–8: An equivalent circuit for an inductor.

In addition to resistance the inductor also has capacitance between each turn and groups of turns. This distributed capacity can approximated by a single capacitor across the entire equivalent circuit giving the equivalent circuit shown in Fig. 1–8.

[1]The value of R_s takes into account the effects of skin effect on wire loss, any losses due to radiation from the inductor, and losses due to dielectrics in the electric field around the inductor. Therefore its calculation is very difficult. If the resistor accounting for the loss is considered to be in parallel with the inductor, then $Q = R_p/(\omega \cdot L)$.

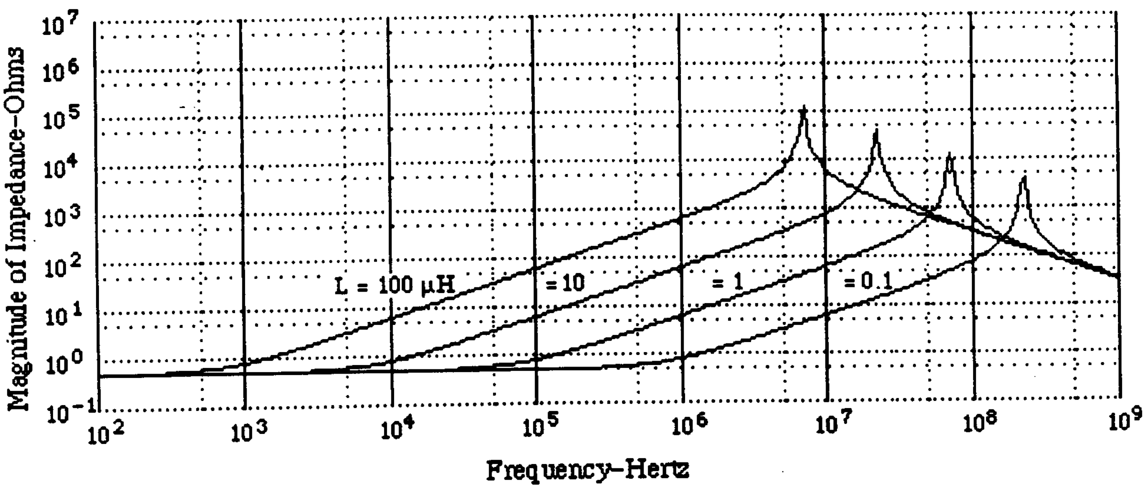

Fig. 1–9: The impedance of a typical inductor with frequency. [$R_s = 0.5\ \Omega$, $C_p = 5$ pF]

The impedance and phase of a typical inductor versus frequency are shown in Fig. 1–9 for several values of inductance, but the same series resistance [$R_s = 0.5\ \Omega$] and parallel capacitance [$C_p = 5$ pF.] In Fig. 1–9 the rising straight line is shows the region where the inductor is acting like it is supposed to. Note at low frequencies the real inductor becomes purely resistive—probably not as badly as shown because the d–c resistance is far less than R_s at 10 MHz. The distributed capacitance which is assumed to be 5 pF causes a parallel resonance at the frequencies where the impedance has a maximum. Note that the effect of C is to increase the effective inductance below the resonant frequency. Far above the resonance the impedance of the "inductor" is essentially determined by the shunting capacitance. If the coil has a magnetic core, it usually will determine the shape of the Q vs. frequency curve.

Note that some inductors are used to carry d–c currents to parts of a circuit. These are usually called r–f chokes. Such a choke is often needed to carry d–c bias voltages to a transistor. In the operating frequency range a choke should provide a high impedance, although not necessarily inductive, so as not to interfere with the operation of the rest of the circuit. Knowledge of a typical impedance curve as in Fig. 1–9 should help design such a choke. The stability of an air–core inductor is primarily determined by its physical dimensions. Therefore winding the coil on a dimensionally stable form is helpful, and the wire should be under some tension so that it does not move when the temperature changes.

Inductors need not be air–cored even at high frequencies. Various magnetic materials may be used to increase the inductance obtained for a given physical size and number of turns. The materials chosen must minimize losses due to hysteresis and eddy currents in the core material. In the range of 10 KHz to about 30 MHz powdered iron cores may be used. Ferrites are usually better and may be obtained with a variety of properties that allow them to be selected for the frequency range in question. The proper ferrite can give good results from audio frequencies through hundreds of MHz. In general, the ferrites useful at high frequencies do not have large permeabilities, however.

An interesting problem that is caused by parasitic elements is encountered when one tries to

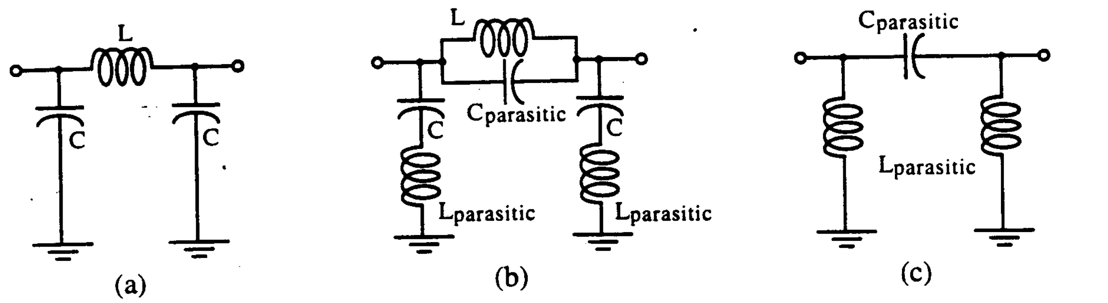

Fig. 1–10: Effect of parasitic elements on a *presumed* lowpass filter structure. (a) The circuit as designed. (b) The circuit with the parasitic elements added. [Resistances are not shown since they do not affect the argument.] (c) The effective circuit at frequencies above the self–resonant frequencies of the elements L and C.

build a low pass filter useful over very wide bands. Such an application might be filtering the power supply leads to an oscillator that must be very well isolated from the outside environment. The normal solution is shown in Fig. 1–10a where an ordinary LC lowpass filter is shown. The selection of proper elements is crucial, however, as is shown in the Fig. 1–10b where the equivalent circuit is shown including the parasitic elements for the L's and C's (The loss resistors are not included because the problem will become evident without them.) If the frequency to be rejected is above the resonant frequencies of the elements, the equivalent circuit becomes as shown in Fig. 1–10c. You will note the circuit has become a highpass filter now, and the rejection of these high frequencies will be extremely poor.

To avoid the problem we must operate at frequencies below the resonance of the elements of the filter. This may be done by raising the resonant frequency of individual elements–in the case of the capacitors–by using types especially designed to have low inductance to ground. Certain feed through and disc capacitors work well. The inductors should be wound to give low distributed capacitance. Too great an inductance is not helpful because of the attendant higher parallel capacitances. The use of ferrite cores helps because the number of turns can be reduced for a given L resulting in a lower distributed C (if one is careful.)

1–1.4 INDUCTIVELY COUPLED CIRCUITS

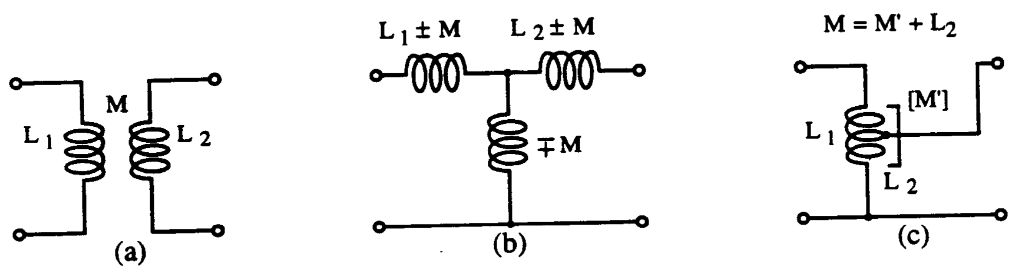

Fig. 1–11: Various forms of inductively coupled circuits. (a) The prototype of two coils coupled entirely by their mutual magnetic fields. (b) An equivalent where the coupling is provided by the circuit. (c) An autotransformer. The coefficient of coupling in all cases is $k = \frac{M}{\sqrt{L_1 \cdot L_2}}$

Figures 1–11a, b, and c show three forms of inductively coupled circuits which can be equivalent. The most common is Fig. 1–11a where the coupling is all by the magnetic field represented by the mutual inductance (M). Depending upon which way the output is taken from the output terminals, M will be positive or negative. The inductances L_1 and L_2 are 'open circuit' inductances; e.g., L_1 is the inductance measured at the left terminals with the right terminals open. At the higher frequencies this may be difficult to do because of the inevitable capacitance across each pair of terminals. The circuit of Fig. 1–11b is exactly equivalent to that for 11a and has the advantage that can be realized without mutual inductance which is relatively hard to calculate and physically realize accurately. For all the inductors to be realizable, the lower signs must be used, and $L_1 - M > 0$ and $L_2 - M > 0$ are required. A third form often used is the auto–transformer. Here M is partly due to the mutual inductance between the upper part of the transformer $(L_1 - L_2)$ and the lower part (L_2), and part due to L_2 itself since both the primary and secondary current flow through L_2. The effective M between input and output is $M = M' + L_2$ where M' is the mutual inductance between the upper and lower portions of the transformer. The auto–transformer is useful in many applications because the tap is easy to place accurately and relatively large values of coupling are easily obtained.

Another factor often used to describe the coupling is the coupling coefficient defined as the ratio of total flux generated by a current in L_1 to that part of the flux which links L_2. The value of k is:

$$k = \frac{M}{\sqrt{L_1 \cdot L_2}} \qquad [1\text{–}8]$$

and $k \leq 1.0$ because of the above definition. In an 'ideal' transformer $k = 1.0$, and both L_1 and L_2 are infinite. The ratio of L_1 to L_2 is finite, however, and equal to the turns ratio squared. At frequencies where a Q–Meter is useful (0.1—50 MHz) the following equation for k is useful:

$$k = \sqrt{1 - \frac{C_{oc}}{C_{sc}}} \qquad [1\text{–}9]$$

where C_{oc} is the capacitance required to resonate [at an arbitrary frequency ω_1] the primary (or secondary) with the secondary (or primary) open–circuited. Likewise C_{sc} is the capacitance required to resonate at the same ω_1 the same side of the transformer with the other side shorted. This method is particularly effective when k is small as is often the case because the Q–Meter is capable of measuring a very small (<1 pF) change of resonating capacitance caused by the open and short circuit conditions.

1–1.5 Measurement of Inductance

At low frequencies inductors can be measured on a variety of low– and radio–frequency bridges and by devices which give a direct reading of inductance. (One such is the LC Meter of Tektronix.) Low frequency measurements of inductors for use at high frequencies tend to give rather incorrect results for L since the effect of the parasitic capacity is minimized in the l–f measurement. The measurement of Q at low frequencies is worse than useless in predicting the high–frequency Q; therefore techniques that measure the inductor at the frequency of interest are necessary if accurate values of L and Q are desired.

An example on one simple instrument which can measure both L and C at radio frequencies is the Q–Meter. The fundamental circuit is as shown in Fig. 1–12: the instrument consists of a low–voltage, very low impedance r–f source variable over a wide range of frequencies, a precisely calibrated variable air capacitor, and an r–f voltmeter. V_1, L, C, and the R_s of the inductor form a series circuit with a loop current of V_1/R_s when the circuit is tuned to resonance $[X_L = X_C]$. The

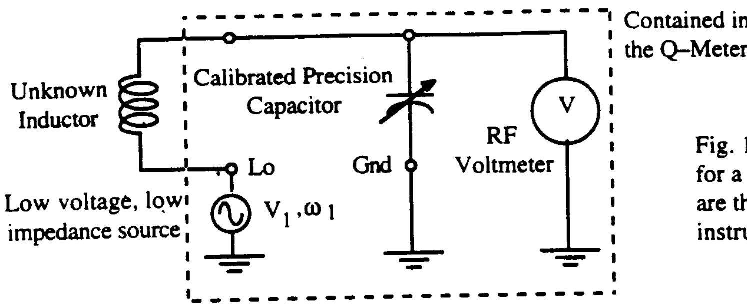

Fig. 1–12: The basic circuit diagram for a Q–Meter. The terminals shown are the external terminals of the instrument.

voltage across C at resonance is then

$$|V_{cap}| \quad = \frac{V_1 \cdot X_C}{R_s} = \frac{V_1 \cdot X_L}{R_s} = V_1 \cdot Q \qquad [1\text{–}10]$$

The reading of the voltmeter is thus proportional to Q. To operate the instrument the inductor to be measured is placed as shown in Fig. 1–12, the desired frequency set, and the capacitor adjusted for resonance which is indicated by a maximum on the meter as the capacitor is changed. From the resulting C the inductance may be calculated.[1] Since the measurement may be made at the operating frequency, the value of Q is valid, but the value of L is the apparent inductance at the given frequency since C_p is in parallel with the measuring capacitor. The error is usually small, but becomes large as the self–resonant frequency of the inductor is approached.

The instrument may be used also to measure small capacitors by first resonating an arbitrary external inductance preferably one with high Q to increase the resolution of the measurement. After noting the capacitance required to resonate the inductor, the unknown C is placed in parallel with the inductor and the decrease in the capacitance of the standard capacitor is equal to the capacitance of the unknown. Such a measurement is known as a substitution measurement since one C is being substituted for another. High resistances may be similarly be measured by noting the change in Q as the resistor is put in parallel with the external inductor. As an example the shunting capacitance of a common resistor may be measured by these techniques: shunting the external inductor with a high resistance changes both the Q and the C required to keep the resonant frequency constant. The ΔC is the shunting capacitance of the resistor.

At any frequency but especially at frequencies above the range of the Q–Meter [>50 MHz] all circuit elements may be measured with transmission test sets or in particular by measuring the S–parameters[2] of the unknown. Such measurements are inherently more difficult and require expensive equipment, but may be done at extremes of frequency. Note that measurements tend to be most accurate near the design impedance of the system which is 50 Ω for most S–parameter test systems; therefore the measurement of a very high impedance inductor will not be extremely accurate with such systems.

SECTION 1–2 FUNDAMENTALS OF TRANSMISSION LINES

1–2.1 THE CHARACTERISTIC IMPEDANCE–Z_0

Transmission lines occur in many forms such as two wire, coaxial, microstrip lines, etc. We

[1]By using certain specific frequencies as marked on the frequency dial, the dial of the capacitor may be calibrated to read not only the capacitance, but also the inductance. The instrument then becomes direct reading.

[2]See Sect. 1–3.

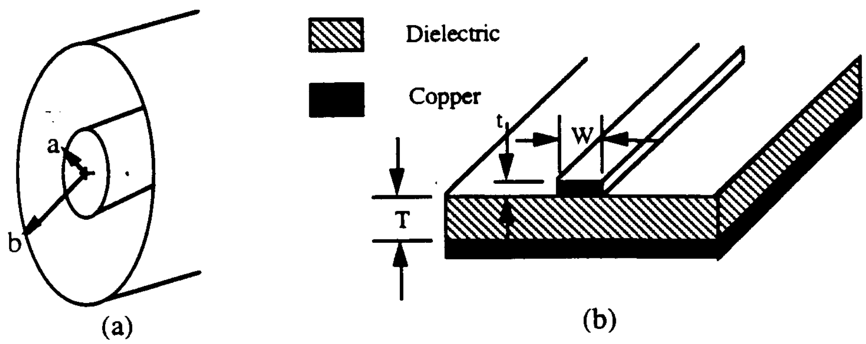

Fig. 1–13: Geometry of a coaxial line (a) and a microstrip transmission line (b).

are primarily concerned with the latter two. The characteristic impedance (Z_0) of a transmission line is that impedance which placed at the receiving end of the line causes the same impedance to appear at the sending end of the line—this is also the sending end impedance of the same line if it were infinitely long. The value of Z_0 in terms of the impedances and admittances per unit length is:

$$Z_0 = \sqrt{\frac{R + j\cdot\omega\cdot L}{G + j\cdot\omega\cdot C}} \qquad [1\text{–}11]$$

[R and L are the series components of the line, and G and C are the shunt components of the line admittance.] If the loss of the line is small per wavelength this equation reduces to the more familiar equation:

$$Z_0 = \sqrt{\frac{L}{C}} \qquad [1\text{–}12]$$

For a coaxial line as illustrated in Fig. 1–13a the characteristic impedance in terms of the dimensions is:

$$Z_0 = \frac{138}{\sqrt{\varepsilon_r}}\cdot \mathrm{Log}_{10}\left(\frac{b}{a}\right) \qquad [1\text{–}13]$$

Where ε_r is the relative dielectric constant of the material filling the line. For the microstrip line (Fig. 1–13b) various equations are given for the Z_0 which are purported to give varying degrees of accuracy. An equation which is good to within a few percent is: (Ref. 2, p. 78)

$$Z_0 = \frac{377\cdot T}{\sqrt{\varepsilon_r}\cdot W}\cdot\frac{1}{1 + 1.735\cdot\varepsilon_r^{-0.0724}\cdot (W/T)^{-0.836}} \qquad [1\text{–}14]$$

Where W is the width of the conductor, and T is the thickness of the dielectric. If exceptional accuracy is required, the non–zero thickness of the conductor must be allowed for by substituting the effective strip width W_{eff} for the actual width W:

$$W_{eff} = W + \frac{t}{\pi}\cdot\left(\mathrm{Log}_e\left(\frac{2\cdot T}{t}\right) + 1\right) \qquad [1\text{–}15]$$

t is the thickness of the conductor. (For an 1/16" board with a 2 mil thick conducting layer the correction is only 0.0033"—less than the precision you'll get cutting the strip.) There is a program on the Macintosh for quickly calculating either Z_0 or the width for a given Z_0 for microstrip lines.

1–2.2 THE PROPAGATION CONSTANT

For a general transmission line the complex propagation constant is:

$$\gamma = \sqrt{(R + j\cdot\omega\cdot L)\cdot(G + j\cdot\omega\cdot C)} = \alpha + j\beta \quad [= j\cdot\sqrt{L\cdot C} \quad \text{if R, G = 0}] \qquad [1\text{–}16]$$

where α is the attenuation constant in nepers per unit length and β is the phase constant in radians per unit length. The wavelength on the line is:

$$\lambda = \frac{2\cdot\pi}{\beta} = \frac{v}{f} \qquad [1\text{–}17]$$

where v = velocity of light *in the line* and f = frequency in Hertz. The velocity of propagation on the line is:

$$v = \frac{\omega}{\beta} = \frac{2\cdot\pi\cdot f}{\beta} \qquad [1\text{–}18]$$

If the losses per wavelength are small, as is usually the case for us with a coaxial lines, the velocity is:

$$v = \frac{c}{\sqrt{\varepsilon_r\cdot\mu_r}} \qquad [c = 3\cdot 10^{10}\ \text{cm/s}] \qquad [1\text{–}19]$$

where ε_r is the relative dielectric constant and μ_r is the relative permeability of the dielectric in the line. Since $\mu_r = 1$ in the usual coaxial line, the velocity is given by:

$$v = \frac{c}{\sqrt{\varepsilon_r}} \qquad [1\text{–}20]$$

The most common dielectric for flexible coaxial lines is polyethylene with $\varepsilon_r = 2.26$; therefore the velocity of propagation in such a line is $2\cdot 10^{10}$ cm/s.

In the microstrip line the situation is more complicated because part of the electric field is in the dielectric and part is in the air. An equation from Ref. 2 for the line wavelength is:

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}}\cdot\sqrt{\frac{\varepsilon_r}{1 + 0.63\cdot(\varepsilon_r - 1)\cdot(W/T)^{0.1255}}} \qquad [W/T > 0.6] \qquad [1\text{–}21]$$

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}}\cdot\sqrt{\frac{\varepsilon_r}{1 + 0.6\cdot(\varepsilon_r - 1)\cdot(W/T)^{0.0297}}} \qquad [W/T < 0.6] \qquad [1\text{–}22]$$

where λ_0 = the free space wavelength for a TEM wave = c/f.

The attenuation constant depends both upon the type of dielectric and the loss in the conductors. In general α increases as $\sqrt{f}$ because of skin effect. (See the references.) One interesting result is that for a line with low dielectric loss (e.g., an air line) the optimum Z_0 to give low loss is 77 Ω. This is one reason most coaxial lines have Z_0 in the range of 50 to 100 ohms.

1–2.3 IMPEDANCE TRANSFORMATIONS BY A TRANSMISSION LINE

Consider a transmission line with a characteristic impedance Z_0 (assumed resistive for our cases which are typical of low loss lines) which is energized at the sending end. When the line is first energized a wave with magnitude E_i travels down the line towards the load. [The subscript **i** means *incident* and the subscript **r** means *reflected*.] The current associated with this wave is I_i =

E_i/Z_0 and is independent of the load. When the wave reaches the load it is absorbed if $Z_L = E_i/I_i = Z_0$ because then all of the energy of the incident wave is absorbed and none remains to be reflected. However, if $Z_L \neq Z_0$), then a wave is reflected as shown in Fig. 1–14. At the end of the line, the boundary condition is:

$$Z_L = \frac{E_i + E_r}{I_i - I_r} = \frac{E_i}{I_i} \cdot \left(\frac{1 + E_r/E_i}{1 - I_r/I_i}\right) \qquad [1\text{–}23]$$

[Note that $E_L = E_i + E_r$ but that $I_L = I_i - I_r$] Therefore—

$$Z_L = Z_0 \cdot \frac{1 + \Gamma}{1 - \Gamma} \qquad [1\text{–}24]$$

where Γ is the reflection coefficient (complex in general) and is defined as:

$$\Gamma = \cancel{\frac{E_i}{I_i}} \frac{E_r}{E_i} \qquad [1\text{–}25]$$

If we solve these equations for Γ in terms of Z_0 and Z_L we get:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \rho\underline{/\theta} \qquad [1\text{–}26]$$

where ρ is the magnitude of the reflection coefficient and θ is its angle. Note that for a lossless line the magnitudes of E_i and E_r are everywhere the same so that ρ is constant anywhere on the line, but the angle θ changes. The angle at any position back from the load a distance of x is:

$$\Gamma(x) = \Gamma(0) \cdot \varepsilon^{-j \cdot 2 \cdot \beta \cdot x} \qquad [1\text{–}27]$$

where $\Gamma(0)$ is the reflection coefficient at the load and $\Gamma(x)$ is the coefficient at the position x units from the load. The value of $\beta \cdot x$ may more conveniently be given in terms of distance and the wavelength of the transmission system:

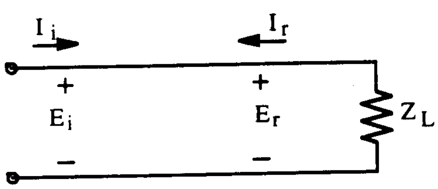

Fig. 1–14: The definitions for the incident and reflected voltage and current waves on a transmission line. [The subscript i indicates incident, r indicates reflected.]

$$\beta \cdot x = \frac{2 \cdot \pi \cdot x}{\lambda} \quad \text{[radians]} \qquad [1\text{–}28]$$

where λ is the wavelength on the line. Note that the phase angle is changed by *twice* the electrical length between the point of measurement and the end of the line.

The impedance, Z_i, looking to the right into the line, a distance x from the termination is:

$$Z_i = Z_0 \cdot \frac{Z_L \cdot \cos(\beta \cdot x) + j \cdot Z_0 \cdot \sin(\beta \cdot x)}{Z_0 \cdot \cos(\beta \cdot x) + j \cdot Z_L \cdot \sin(\beta \cdot x)} \qquad [1\text{–}29]$$

$$Z_i = Z_0 \cdot \frac{Z'_L + j \cdot \tan(\beta \cdot x)}{1 + j \cdot Z'_L \cdot \tan(\beta \cdot x)} \qquad [1\text{–}30]$$

$$Z'_L = Z_L/Z_0 \qquad [1\text{–}31]$$

where $\beta \cdot x$ is the electrical length of the line from the point of measurement to the termination.

Special Case — $x = \lambda/4$ — **Quarter Wave Line**

If the line is a quarter–wave long, i.e., $x = \lambda/4$ [or $\theta = \pi/2$], then $\cos(\beta \cdot x) = 0$ and $\sin(\beta \cdot x) = 1$; therefore:

$$Z_i = \frac{Z_0^2}{Z_L} \qquad [1-32]$$

A line used this way acts like an impedance transformer. For example if $Z_L = 100$ ohms and $Z_0 = 70.7$ ohms, the input impedance of the quarter–wave line is:

$$Z_i = 70.7^2/100 = 50\ \Omega$$

Such a line would be a good way of matching the 100 ohm load to a standard 50 ohm line (but only over a rather narrow band of frequencies.)

A common practice is to normalize all impedances to Z_0; i.e., $Z' = Z/Z_0$ and $Y' = Y/Y_0 = Y \cdot Z_0$. For normalized impedances the quarter–wave line acts as an impedance inverter:

$$Z_i' = 1/Z_L' \qquad [1-33]$$

As an example, suppose a $-j50$ capacitor is the load [$Z_L' = -j$]; then:

$$Z_i' = 1/(-j) = +j \qquad [1-34]$$

and the capacitive load is transformed into an inductive impedance—equal to $+j50\ \Omega$.

Special Case — $x = \lambda/2$ — **Half Wave Line**

For a half–wave line $\theta = \pi$ and $\cos(\beta \cdot x) = -1$, $\sin(\beta \cdot x) = 0$; therefore the input impedance is the same as the load impedance.Note that the impedance looking into a transmission line repeats every $\lambda/2$ along the line.

Special Case — $Z_L = 0$ — **Shorted Load**

For the case that $Z_L = 0$ the input impedance is a pure reactance:

$$Z_i = +j \cdot \tan(\beta \cdot x) = +j \cdot \tan(2 \cdot \pi \cdot x/\lambda) \qquad [1-35]$$

For $x < \lambda/4$ the impedance obtained is inductive and for $\lambda/4 < x < \lambda/2$ the impedance is capacitive. If the line losses are low, this shorted line is the basis of a high–Q resonator. Such a line is also used in matching networks to provide a known amount of inductive reactance. This reactance may be approximated if the line length is small ($x/\lambda < 0.01$) since Tan (θ) $\cong \theta$ for small θ.

$$X_L = Z_0 \cdot \tan(\beta \cdot x) \cong Z_0 \cdot \beta \cdot x \cong Z_0 \cdot \frac{2 \cdot \pi \cdot f \cdot x}{v} \qquad [1-36]$$

where $v =$ velocity of propagation on the line and x is the distance to the short. Since $X_L = 2 \cdot \pi \cdot f \cdot L$, where L is the inductance looking into the line, we may write:

$$L \cong \frac{Z_0 \cdot x}{v} \qquad (x/\lambda < 0.01) \qquad [1-37]$$

Special Case — $Z_L = \infty$ — **Open Circuited Load**

For the case $Z_L = \infty$ the input impedance becomes:

$$Z_i = -j \cdot Z_0 \cdot \text{ctn}(\beta \cdot x) = -j \cdot Z_0 \cdot \text{ctn}(2 \cdot \pi \cdot x/\lambda) \qquad [1-38]$$

If, as is usually the case, $\beta \cdot x < \pi/2$ [$x < \lambda/4$], the impedance is capacitive. The capacitance so formed can be used in ways similar to the preceding example. For short lines ($x/\lambda < 0.01$) we can again approximate the tangent by its angle giving:

$$X_C = \frac{Z_0}{\tan(\beta \cdot x)} \cong \frac{Z_0}{\beta \cdot x} \cong \frac{v \cdot Z_0}{2 \cdot \pi \cdot f \cdot x} \qquad [1\text{–}39]$$

Since $X_C = 1/(2 \cdot \pi \cdot f \cdot C)$, we can solve for C:

$$C \cong \frac{x}{v \cdot Z_0} \qquad (x/\lambda < 0.01) \qquad [1\text{–}40]$$

This approximate value of C is useful in calculating matching networks also. The approximation is perfect at zero frequency and has infinite error where the line segment is 1/4 wave long.

SECTION 1–3 CHARACTERIZATION OF NETWORKS BY TRANSMISSION PARAMETERS

1–3.1 S–PARAMETERS

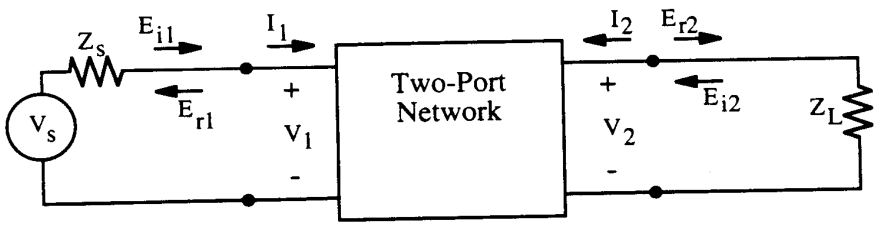

Fig. 1–15: The definitions of currents an voltages in describing a two–port network. In addition the incident and reflected travelling waves [E_i and E_r] are also shown.

The characterization of the two–port network shown in Fig. 1–15 may be accomplished by a number of parameter sets, for example the h, y, and z parameter sets. The defining equations are:

h–Parameters

$$V_1 = h_{11} \cdot I_1 + h_{12} \cdot V_2 \qquad [1\text{–}41]$$

$$I_2 = h_{21} \cdot I_1 + h_{22} \cdot V_2 \qquad [1\text{–}42]$$

y–Parameters

$$I_1 = y_{11} \cdot V_1 + y_{12} \cdot V_2 \qquad [1\text{–}43]$$

$$I_2 = y_{21} \cdot V_1 + y_{22} \cdot V_2 \qquad [1\text{–}44]$$

z–Parameters

$$V_1 = z_{11} \cdot I_1 + z_{12} \cdot I_2 \qquad [1\text{–}45]$$

$$V_2 = z_{21} \cdot I_1 + z_{22} \cdot I_2 \qquad [1\text{–}46]$$

The only difference in these parameter sets is the choice of independent and dependent variables. The parameters are the constants used to relate these variables. In general four parameters are required for a linear, non–time varying two port. [Actually there are eight constants since each is a complex number.]

However, when it comes to measurement, the parameter sets differ considerably in their practical ease of measurement. In principle any four independent measurements can be used to determine the four parameters—for example measurements of both voltages and both currents with four different load impedances could be used, but the accuracy obtained would usually be poor, and the computations difficult. Usually the measurement is attempted in a way to eliminate one of the currents or voltages so that a relatively direct measurement of a pair of parameters is obtained. As an example, if the output of the box is shorted then $V_2 = 0$ and h_{11} and h_{21} are easily obtained:

$$h_{11} = V_1/I_1 \qquad [V_2 = 0] \qquad [1\text{–}47]$$

$$h_{21} = I_2/I_1 \qquad [V_2 = 0] \qquad [1\text{–}48]$$

Likewise, if we open circuit the input making $I_1 = 0$ and drive the output terminal, we get:

$$h_{12} = V_1/V_2 \qquad [I_1 = 0] \qquad [1\text{–}49]$$

$$h_{22} = I_2/V_2 \qquad [I_1 = 0] \qquad [1\text{–}50]$$

With the Y–Parameters the easiest terminations are short circuits, and with the Z–Parameters the easiest terminations are open circuits. In principle these parameter sets may be used at any frequency, and in fact, for some calculations they are useful at high frequencies; however, some problems arise at high frequency in the *measurement* of the parameters:

1. Equipment is not readily available to measure total voltage and total current at the ports of the network.
2. Precise short and open circuits located at the terminals of the two–port are difficult to achieve over broad bands of frequencies.
3. Active devices such as transistors may not be stable (i.e., they may oscillate) when open– or short–circuit terminated.

Quantities that can be easily measured at high frequencies are waves on transmission lines and ratios of incident and reflected waves. The concept of defining impedances in a transmission line system by the combination of an incident and a reflected wave was discussed in a previous section. The total voltage at a given point of the line is the sum of the incident and reflected voltage waves:

$$V_t = E_i + E_r \qquad [1\text{–}51]$$

The current at a point in the line is the difference in the incident and reflected current waves:

$$I_t = I_i - I_r = E_i/Z_0 - E_r/Z_0 \qquad [1\text{–}52]$$

Also we defined a reflection coefficient as:

$$\Gamma = E_r/E_i = \rho\underline{/\theta} \qquad [1\text{–}53]$$

$$= \frac{Z_L - Z_0}{Z_L + Z_0} \qquad [1\text{–}54]$$

If we generalize these transmission line concepts to a two port network as shown in Fig. 1–15 we can define the network by the two waves incident upon the network, E_{i1} and E_{i2}, and the two reflected waves, E_{r1} and E_{r2}. The h–parameters are defined in terms of V_1, V_2, I_1, and I_2. Since we also have equations for these voltages and currents in terms of the incident and reflected voltage waves we can express the parameters in terms of the waves. For example:

$$V_1 = h_{11}\cdot I_1 + h_{12}\cdot V_2 = E_{i1} + E_{r1} \qquad [1\text{–}55]$$

These sets of equations may be solved to find the reflected waves as functions of the incident waves and the h–parameters:

$$E_{r1} = f_{11}(h)\cdot E_{i1} + f_{12}(h)\cdot E_{i2} \qquad [1\text{–}56]$$

$$E_{r1} = f_{21}(h)\cdot E_{i1} + f_{22}(h)\cdot E_{i2} \qquad [1\text{–}57]$$

The functions f_{11}, f_{12}, f_{21}, and f_{22} are a new set of parameters which relate the travelling voltage waves rather than total voltages and currents. In this case the f's are functions of the h-parameters, but the other parameter sets could be used just as well. Since the f's relate the incident to the reflected (or scattered) wave, the new parameters are called the *scattering (or S–)* parameters. The more usual form is obtained by dividing both sides of the equations by $\sqrt{Z_0}$ so that we have a new set of variables[1]:

$$a_1 = \frac{E_{i1}}{\sqrt{Z_0}} \qquad a_2 = \frac{E_{i2}}{\sqrt{Z_0}} \qquad [1\text{–}58]$$

$$b_1 = \frac{E_{r1}}{\sqrt{Z_0}} \qquad b_2 = \frac{E_{r2}}{\sqrt{Z_0}} \qquad [1\text{–}59]$$

Note that the square of the magnitude of these new variables is the *power* in a particular wave, i.e., $|a_1|^2$ is the incident power on the input port, and $|b_1|^2$ is the power reflected from the input. Using the new parameters the s–parameters relate the travelling waves as follows:

$$b_1 = s_{11}\cdot a_1 + s_{12}\cdot a_2 \qquad [1\text{–}60]$$

$$b_2 = s_{21}\cdot a_1 + s_{22}\cdot a_2 \qquad [1\text{–}61]$$

To see why these new parameters are of advantage let us look at the measurement of them. To measure s_{11} we terminate the output port with the characteristic impedance of the transmission lines used, Z_0. In this way a_2 is made equal to zero *regardless of the length of the transmission lines used.* Solving Eq. 1–60 and 1–61 under these conditions gives:

$$s_{11} = b_1/a_1 = \Gamma_1 \qquad [1\text{–}62]$$

$$s_{21} = b_2/a_1 \qquad [1\text{–}63]$$

Therefore s_{11} is the input reflection coefficient (Γ_1) with the output of the network terminated in Z_0. (Note that the output of the network might, or might not, be matched by Z_0, but the important thing is that the output line is matched by Z_0 so that nothing is reflected from the load.)

To measure the other two S–parameters we drive the network at the output or port 2 and terminate the input line with Z_0. This makes E_{i1} and a_1 zero so that:

$$s_{22} = b_2/a_2 = \Gamma_2 \qquad [1\text{–}64]$$

$$s_{12} = b_1/a_2 \qquad [1\text{–}65]$$

From the preceding you can see the advantages of the S–parameter measurements are: 1) The network is always terminated in lossy impedances (Z_0) which minimize the possibility of oscillation. 2) The desired terminations are independent of line length which is not the case when open or short–circuit loads are required. 3) The measurement of reflection coefficient is relatively easy using, for example, directional couplers.

1–3.2 Summary of S–Parameter Definitions

s_{11} = Input reflection coefficient with the output port terminated in Z_0.

s_{21} = Forward transmission coefficient with Z_0 load.

$|s_{21}|^2$ = Transducer power gain with Z_0 source and load.

s_{22} = Output reflection coefficient with Z_0 source (and $V_s = 0$).

[1]The reason for this particular normalization will become apparent later when the parameter are actually used.

s_{12} = Reverse transmission coefficient with the source impedance = Z_0.

$|s_{12}|^2$ = Reverse transducer power gain with Z_0 source and load.

Some properties of typical networks as expressed by S–parameters are:

Reciprocal Networks

A reciprocal network has identical transmission characteristics in either direction. This implies that the S–parameter matrix is equal to its transpose:

$$s_{12} = s_{21} \qquad [1\text{–}66]$$

Lossless Network

Since a lossless network does not dissipate any power, the power incident on the network must equal the power reflected, i.e., $|a_n|^2 = |b_n|^2$. For this to be true:

$$\mathbf{I} - \mathbf{S}^*\cdot\mathbf{S} = 0 \qquad [1\text{–}67]$$

where **I** is the identity matrix and **S*** is the complex conjugate of the transpose of **S**, or the hermetian conjugate of **S**.

Lossy Network

If the network is lossy

$$\mathbf{I} - \mathbf{S}^*\cdot\mathbf{S} > 0 \qquad [1\text{–}68]$$

(In this case the total reflected power is less than the total incident power.)

[The preceding notes on S–parameters are abstracted from HP Application Note 154, April, 1972] The following pages give useful equations for calculating gains and impedances from S–parameter data. Also the conversion equations from s to y to z to h parameters are given. Since these equations usually involve complex numbers, they are tedious to use in practice without the aid of a computer. Macintosh programs that do many of these calculations for you are available. You should learn how to use these programs because one important thing to do during the quarter is to compare the overall measured amplifier performance with what you compute knowing the transistor and circuit parameters. It is also important in such computations to reconcile the differences between the two approaches, i.e., are the differences due to errors in procedure, to the influence of the measurement on the quantity measured, to unknown parasitic parameters, or to the inherent accuracy of the measuring equipment—are the errors within the bounds of expected measurement accuracy? (Nothing's perfect!)

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TRANSFORMATION OF S– TO OTHER PARAMETERS

Equations for transforming from y-to-s, y-to-h and vice versa are useful and are given in the following. Since these equations usually involve complex numbers, they are tedious in practice to use without the aid of a computer.

$$s_{11} = \frac{(1-y_{11}')\cdot(1+y_{22}') + y_{12}'\cdot y_{21}'}{(1+y_{11}')\cdot(1+y_{22}') - y_{12}'\cdot y_{21}'} \qquad y_{11}' = \frac{(1-s_{11})\cdot(1+s_{22}) + s_{12}\cdot s_{21}}{(1+s_{11})\cdot(1+s_{22}) - s_{12}\cdot s_{21}}$$

$$s_{12} = \frac{-2\cdot y_{12}'}{(1+y_{11}')\cdot(1+y_{22}') - y_{12}'\cdot y_{21}'} \qquad y_{12}' = \frac{-2\cdot s_{12}}{(1+s_{11})\cdot(1+s_{22}) - s_{12}\cdot s_{21}}$$

$$s_{21} = \frac{-2\cdot y_{21}'}{(1+y_{11}')\cdot(1+y_{22}') - y_{12}'\cdot y_{21}'} \qquad y_{21}' = \frac{-2\cdot s_{21}}{(1+s_{11})\cdot(1+s_{22}) - s_{12}\cdot s_{21}}$$

$$s_{22} = \frac{(1+y_{11}')\cdot(1-y_{22}') + y_{12}'\cdot y_{21}'}{(1+y_{11}')\cdot(1+y_{22}') - y_{12}'\cdot y_{21}'} \qquad y_{22}' = \frac{(1+s_{11})\cdot(1-s_{22}) + s_{12}\cdot s_{21}}{(1+s_{11})\cdot(1+s_{22}) - s_{12}\cdot s_{21}}$$

Note: $y' = R_0\cdot y$; i.e., the normalized units are all dimensionless.

$$s_{11} = \frac{(h_{11}'-1)\cdot(1+h_{22}') - h_{12}\cdot h_{21}}{(1+h_{11}')\cdot(1+h_{22}') - h_{12}\cdot h_{21}} \qquad h_{11}' = \frac{(1+s_{11})\cdot(1+s_{22}) - s_{12}s_{21}}{(1-s_{11})\cdot(1+s_{22}) + s_{12}\cdot s_{21}}$$

$$s_{12} = \frac{2\cdot h_{12}}{(1+h_{11}')\cdot(1+h_{22}') - h_{12}\cdot h_{21}} \qquad h_{12} = \frac{2\cdot s_{12}}{(1-s_{11})\cdot(1+s_{22}) + s_{12}\cdot s_{21}}$$

$$s_{21} = \frac{-2\cdot h_{21}}{(1+h_{11}')\cdot(1+h_{22}') - h_{12}\cdot h_{21}} \qquad h_{21} = \frac{-2\cdot s_{21}}{(1-s_{11})\cdot(1+s_{22}) + s_{12}\cdot s_{21}}$$

$$s_{22} = \frac{(h_{11}'+1)\cdot(1-h_{22}') + h_{12}\cdot h_{21}}{(1+h_{11}')\cdot(1+h_{22}') - h_{12}\cdot h_{21}} \qquad h_{22}' = \frac{(1-s_{11})\cdot(1-s_{22}) - s_{12}s_{21}}{(1-s_{11})\cdot(1+s_{22}) + s_{12}\cdot s_{21}}$$

Note: $h_{11}' = h_{11}/R_0$; $h_{12}' = h_{12}$; $h_{21}' = h_{21}$; $h_{22}' = h_{22}\cdot R_0$

S–PARAMETER RELATIONSHIPS

Input reflection coefficient [s_{11}' or Γ_{in}] with arbitrary Z_L

$$s_{11}' = s_{11} + \frac{s_{12}\cdot s_{21}\cdot\Gamma_L}{1 - s_{22}\cdot\Gamma_L} = \Gamma_{in}$$

[Note that if $Z_L = Z_0$ then $\Gamma_L = 0$ and $s_{11}' = s_{11}$.]

Output reflection coefficient [s_{22}'] with arbitrary Z_S

$$s_{22}' = s_{22} + \frac{s_{12} \cdot s_{21} \cdot \Gamma_S}{1 - s_{11} \cdot \Gamma_S}$$

Voltage gain at the terminals of the two-port with arbitrary Z_L and Z_S

$$A_V = \frac{s_{21} \cdot (1 + \Gamma_L)}{(1 - s_{22} \cdot \Gamma_L) \cdot (1 + s_{11}')}$$

$$\text{Available power gain} \triangleq \frac{\text{Power available from the network}}{\text{Power available from the source}}$$

$$G_A = \frac{|s_{21}|^2 \cdot (1 - |\Gamma_S|^2)}{(1 - |s_{22}|^2) + |\Gamma_S|^2 \cdot (|s_{11}|^2 - |D|^2) - 2 \cdot \mathrm{Re}(\Gamma_S \cdot M)}$$

$$D \triangleq s_{11} \cdot s_{22} - s_{12} \cdot s_{21}$$

$$M \triangleq s_{11} - D \cdot s_{11}^*$$

$$\text{Transducer power gain} \triangleq \frac{\text{Power delivered to load}}{\text{Power available from the source}}$$

$$G_T = \frac{|s_{21}|^2 \cdot (1 - |\Gamma_S|^2) \cdot (1 - |\Gamma_L|^2)}{|(1 - s_{11} \cdot \Gamma_S) \cdot (1 - s_{22} \cdot \Gamma_L) - s_{12} \cdot s_{21} \Gamma_L \cdot \Gamma_S|^2}$$