ECE 145A / 218 C, notes set 3: Two-Port Parameters

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Device Descriptions for Circuit Design

Equivalent - circuit model

physically based

includes dependence upon DC bias & frequency

often includes device size dependence

weakness: necessary simplified, hence some errors

2 - Port Model

matrix of tabular data

need one model for each bias point, each frequency

huge data sets required.

medium for both (a) measured data and (b) E/M simulation data

2 - port methods also useful for general network theory.
HBT equivalent-circuit model

\[ R_{be} = \beta / g_m \]

\[ \tau_f = \tau_b + \tau_c \]

\[ \tau_f = \tau_{\text{base}} + \tau_{\text{collector}} \]

\[ g_{mo} \equiv \frac{\partial I_C}{\partial V_{BE}} = \frac{I_C}{(nkT / q)} \]

\[ g_m = g_{mo} e^{-j\omega \gamma c} \quad 0 < \gamma < 1 \quad (\text{typically} \sim 0.8) \]

We will cover this in more detail soon
FET Equivalent-Circuit Model

We will cover this in more detail soon

\[ g_m \approx \frac{\varepsilon}{T_{eq}} v_{eff} (NW_g) \text{ or } \frac{\varepsilon}{T_{eq}} \mu (NW_g) (V_{gs} - V_{th}) \]

\[ C_{gd} \approx k_o W_g \]

\[ k_o \approx (0.3 - 0.5) \text{ fF/\mu m} \]

\[ C_{gs} \approx \frac{\varepsilon}{T_{eq}} L_g (NW_g) + k_o W_g \]

\[ G_{ds} \propto NW_g \]

\[ R_g \sim \frac{\rho_s}{12L_g} \left( \frac{W_g}{N} \right) + \frac{R_{end}}{2N} \]

\[ R_d \propto 1/NW_g \]

\[ R_s \propto 1/NW_g \]

\[ C_{sb} \propto NW_g \]

\[ C_{db} \propto NW_g \]

Increase \( f_{max} \) using - short gate fingers - ample substrate contacts
Box might contain: a transistor, a passive element, a subcircuit

The terminal characteristics relate the variables $V_1$, $V_2$, $I_1$, and $I_2$. There are 2 degrees of freedom.

Any two variables can be set as the *independent variables*. The remaining two variables, the *dependent variables*, can then be written as functions of the independent variables.
Two-Port Parameters: Represent Device or Network
Admittance Parameters

Frequency-domain description:

\[ v_1(t) = \text{Re}\{V_1 e^{i\omega t}\} \quad , \quad i_1(t) = \text{Re}\{I_1 e^{i\omega t}\} \], etc.

\[
\begin{bmatrix}
I_1(j\omega) \\
I_2(j\omega)
\end{bmatrix} =
\begin{bmatrix}
Y_{11}(j\omega) & Y_{12}(j\omega) \\
Y_{21}(j\omega) & Y_{22}(j\omega)
\end{bmatrix}
\begin{bmatrix}
V_1(j\omega) \\
V_2(j\omega)
\end{bmatrix}
\]

Currents are written as functions of voltages.
DC bias is taken as implicit.
Admittance Parameters Example: Simple FET Model

\[
\begin{pmatrix}
I_1 \\
I_2
\end{pmatrix} =
\begin{bmatrix}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{bmatrix}
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix}
\]

By inspection:

\[
Y = [Y_{ij}] =
\begin{bmatrix}
j\omega(C_{gs} + C_{gd}) & -j\omega C_{gd} \\
g_m - j\omega C_{gd} & G_{ds} + j\omega C_{gd}
\end{bmatrix}
\]
Impedance Parameters

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= \begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]

Voltages are written as functions of currents.

Example:

\[
Z = [Z_{ij}] = \begin{bmatrix}
R_1 + R_3 & R_3 \\
R_3 & R_2 + R_3
\end{bmatrix}
\]

...easy!
Hybrid Parameters: Old and Obscure

\[
\begin{bmatrix}
V_1 \\
I_2
\end{bmatrix} =
\begin{bmatrix}
h_{11} & h_{12} \\
h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
V_2
\end{bmatrix}
\]

This is certainly an odd choice of independent variables.
Hybrid Parameter Example: Simple BJT Model

\[
\begin{bmatrix}
V_1 \\
I_2
\end{bmatrix}
= 
\begin{bmatrix}
h_{11} & h_{12} \\
h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
V_2
\end{bmatrix}
\]

note: \( g_m V_{be} = \beta V_{be} / R_{be} = \beta \tilde{I}_b \)

By inspection:

\[
H = \begin{bmatrix}
1 & 0 \\
1 / R_{be} + j \omega C_{be} & 1 \\
1 + j \omega R_{be} C_{be} & 1 / R_{ce}
\end{bmatrix}
\]

This is related to short-circuit current gain.
Short-Circuit Current Gain

short-circuit current gain \( \equiv \frac{I_2}{I_1} \bigg|_{V_2=0} = \frac{I_2}{I_1} \) \( = h_{21} \)

\[ f_\tau = \text{frequency at which } \| h_{21} \| \text{ extrapolates to } 1. \]

\[ = "\text{short-circuit current - gain cutoff frequency}". \]

For the highly simplified model on the prior page, if \( \beta \gg 1 \),

\[ f_\tau \approx g_m / 2\pi C_{be}. \]
Definition of S-parameters... with a's and b's

S-parameters are rigorously defined in terms of the wave amplitudes on transmission lines connected to the device under test:

\[
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  S_{11} & S_{12} \\
  S_{21} & S_{22}
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2
\end{bmatrix}
\]

...where the a's and b's are the wave amplitudes.

\[ a = V^+ / \sqrt{Z_0} \text{ and } b = V^- / \sqrt{Z_0} \]
Definition of S-parameters... with V+'s and V-'s

We can also write

\[
\begin{bmatrix}
V_1^- \\
V_2^-
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
\begin{bmatrix}
V_1^+ \\
V_2^+
\end{bmatrix}
\]

"+" waves travel towards the 2 - port;"-" waves travel away.
De-Mystifying S-Parameters

At port 1, \( V_1 = V_1^+ + V_1^- \) and \( I_1 = \frac{(V_1^+ - V_1^-)}{Z_0} \)

At port 2, \( V_2 = V_2^+ + V_2^- \) and \( I_2 = \frac{(V_2^+ - V_2^-)}{Z_0} \)

If we know the relationship between \([I_1, I_2]\) and \([V_1, V_2]\): (Y or Z parameters),
we can calculate the relationship between \([V_1^+, V_2^+]\) and \([V_1^-, V_2^-]\): (S parameters).
How to Compute S-parameters quickly: $S_{11}$

If $Z_L = Z_0$ then $\Gamma_L = 0$, hence $V_2^+ = 0$

Now $S_{11} = \left. \frac{V_1^-}{V_1^+} \right|_{V_2^+ = 0}$, so $S_{11} = \left. \frac{V_1^-}{V_1^+} \right|_{Z_L = Z_0}$

Defining $Z_{in}\bigg|_{Z_L = Z_0}$ as the input impedance given $Z_L = Z_0$,

$$S_{11} = \frac{Z_{in}\bigg|_{Z_L = Z_0} - Z_0}{Z_{in}\bigg|_{Z_L = Z_0} + Z_0}$$
Input impedance = Input Reflection Coefficient

Noting that \( S_{11} = \frac{Z_{in}|_{Z_L=Z_0} - Z_0}{Z_{in}|_{Z_L=Z_0} + Z_0} \)

Note that reflection coefficient \( S_{11} \) is a method of specifying input impedance.

\[
\begin{bmatrix}
S_{11} \\
Z_{in}|_{Z_L=Z_0}
\end{bmatrix}
\text{ is the input to the system.}
\begin{bmatrix}
\text{reflection coefficient} \ S_{11} \\
\text{impedance}
\end{bmatrix}
\]

given that the load \( \begin{bmatrix}
\text{reflection coefficient} \ S_{11} \\
\text{impedance}
\end{bmatrix} \) is \( \begin{bmatrix}
0 \\
Z_0
\end{bmatrix} \).
Output impedance = Output Reflection Coefficient

The same analysis & comments clearly applies to $S_{22}$.

By symmetry, $S_{22} = \frac{Z_{out}|_{Z_{gen}=Z_0} - Z_0}{Z_{out}|_{Z_{gen}=Z_0} + Z_0}$

Note that reflection coefficient $t(S_{22})$ is a method of specifying output impedance.

\[
\begin{align*}
\begin{cases}
S_{22} \\
Z_{out}|_{Z_{gen}=Z_0}
\end{cases}
\end{align*}
\]

is the output

\[
\begin{cases}
\text{reflection coefficient } t
\end{cases}
\]

impedance

given that the generator

\[
\begin{cases}
\text{reflection coefficient } t
\end{cases}
\]

impedance

is \[
\begin{cases}
0 \\
Z_0
\end{cases}
\]
Computing $S_{11}$: Example

Given $Z_0 = 50\Omega$, what is $S_{11}$?

$Z_{in}\big|_{Z_L=Z_0} = 54\Omega$

So

$$S_{11} = \frac{Z_{in}\big|_{Z_L=Z_0} - Z_0}{Z_{in}\big|_{Z_L=Z_0} + Z_0} = \frac{54\Omega - 50\Omega}{54\Omega + 50\Omega} = \frac{4}{104}$$

by similar arguments, $S_{22} = 4/104$
Computing $S_{21}$

Set: $Z_{\text{gen}} = Z_L = Z_0$

Given $Z_{\text{gen}} = Z_0$ we have $\Gamma_s = 0$ and $T_s = Z_0/(Z_0 + Z_{\text{gen}}) = 1/2$,

hence $V_{1}^+ = T_s V_{\text{gen}} + \Gamma_s V_{1}^- = V_{\text{gen}} / 2$.

Given $Z_L = Z_0$ we have $\Gamma_L = 0$, hence $V_{\text{out}} = V_2 = V_{2}^+ + V_{2}^- = V_{2}^-$

So $S_{21} = \left| \frac{V_2^-}{V_1^+} \right|_{V_2^+=0}^{Z_L=Z_0} = \left| \frac{V_2^-}{V_1^+} \right|^{Z_L=Z_0} = \left| \frac{V_2^-}{V_1^+} \right|^{Z_L=Z_0} = \left| \frac{V_{\text{out}}}{V_{\text{gen}} / 2} \right|^{Z_L=Z_0=Z_{\text{gen}}} = \left| \frac{2V_{\text{out}}}{V_{\text{gen}}} \right|^{Z_L=Z_0=Z_{\text{gen}}}$
Relating amplifier Gains to S-parameters...$S_{21}$

These relationships allow us to develop a simpler way of finding the $S$-parameters:

$$S_{21} = 2 \frac{v_{out}}{v_{gen}} \left| \begin{array}{c} \text{generator} = \text{load} = Z_0 \end{array} \right.$$  

..which is simply **how much bigger** *the signal became upon* *insertion* *of the amplifier* in the 50 Ohm system. $S_{21}$ is called *the* *insertion gain*. 

\[ Z_0 \]
Relating amplifier Gains to S-parameters... $S_{12}$

By symmetry

$$S_{12} = 2 \frac{v_{in}}{v_{gen}} \bigg|_{Z_{generator}=Z_{load}=Z_0}$$

Diagram showing a circuit with $R_{gen} = Z_0$, $V_{in}$, $V_{gen}$, and $R_{L} = Z_0$. The equation for $S_{12}$ is derived using the symmetry of the circuit.
S11 and S22 can be directly related to input and output impedances

\[
S_{11} = \frac{(Z_{in} / Z_0) - 1}{(Z_{in} / Z_0) + 1}, \text{ where } Z_{in} = Z_{in} \left|_{Z_{load} = Z_0} \right.
\]

\[
S_{22} = \frac{(Z_{out} / Z_0) - 1}{(Z_{out} / Z_0) + 1}, \text{ where } Z_{out} = Z_{out} \left|_{Z_{generator} = Z_0} \right.
\]

...in practice, we do not need to plug into the formulas: knowing the \(Z_{in}\) tells us the \(S_{11}\), because the formula is one-to-one and is neatly represented by the Smith chart
Relating amplifier Gains to S-parameters... S11 and S22

\[ S_{11} = \frac{(Z_{in} / Z_0) - 1}{(Z_{in} / Z_0) + 1}, \text{ where } Z_{in} = Z_{in} \bigg|_{Z_{load} = Z_0} \]

\[ S_{22} = \frac{(Z_{out} / Z_0) - 1}{(Z_{out} / Z_0) + 1}, \text{ where } Z_{out} = Z_{out} \bigg|_{Z_{generator} = Z_0} \]

we do not need to plug into the formulas: the Smith chart is a plot of this formula, so the Smith chart plots \( S_{ij} \) and \( Z_{in} \) at the same time
Example of working with S-parameters

\[ S_{21} = \frac{-2g_m Z_o}{1 + j\omega C_{in} Z_o} \]

\[ S_{11} = \frac{(Z_{in} / Z_0) - 1}{(Z_{in} / Z_0) + 1} \], where \( Z_{in} = 1 / j\omega C_{in} \)

\[ S_{22} = \frac{(Z_{out} / Z_0) - 1}{(Z_{out} / Z_0) + 1} \], where \( Z_{out} = \) infinity

\[ S_{12} = 0 \]

...easy !!!
Example of working with S-parameters

\[
S_{21} = \frac{\left( \begin{array}{c} \frac{1}{Z_0} \\ j\omega C \end{array} \right)}{\left( \begin{array}{c} \frac{1}{Z_0} \\ j\omega C \end{array} \right) + Z_0} = S_{12} = \frac{1}{1 + j\omega C Z_0 / 2}
\]

\[
S_{11} = \left( \frac{Z_{in} / Z_0}{Z_{in} / Z_0} + 1 \right) - 1 = \frac{j\omega C Z_0 / 2}{1 + j\omega C Z_0 / 2}, \text{ where } Z_{in} = \left( \begin{array}{c} \frac{1}{Z_0} \\ j\omega C \end{array} \right)
\]

\[
S_{22} = \left( \frac{Z_{out} / Z_0}{Z_{out} / Z_0} + 1 \right) - 1, \text{ where } Z_{out} = \left( \begin{array}{c} \frac{1}{Z_0} \\ j\omega C \end{array} \right)
\]

...this illustrates the importance of "\( Z_{in} \mid_{Z_L=Z_o} \)", etc
Why do we care about impedances matched to 50 Ohms?

Standing waves on transmission lines cause gain/phase ripples of the form \((1 - \Gamma_S \Gamma_L \exp(-j2f\tau))^{-1}\). Either we must have short transmission lines, or the lines must be well-terminated.