ECE 145B / 218B, notes set 5: Two-port Noise Parameters

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References and Citations:

Sources / Citations:
Kittel and Kroemer : Thermal Physics
Van der Ziel : Noise in Solid - State Devices
Papoulis : Probability and Random Variables (hard, comprehensive)
Wozencraft & Jacobs : Principles of Communications Engineering.
Motchenbaker : Low Noise Electronic Design
Information theory lecture notes : Thomas Cover, Stanford, circa 1982
Probability lecture notes : Martin Hellman, Stanford, circa 1982
National Semiconductor Linear Applications Notes : Noise in circuits.

Suggested references for study.
Van der Ziel, Wozencraft & Jacobs, Peebles, Kittel and Kroemer
Papers by Fukui(device noise), Smith & Personik (optical receiver design)
National Semi. App. Notes (!)
Cover and Williams : Elements of Information Theory
Our Notation for Spectral Densities and Correlations

<table>
<thead>
<tr>
<th>Random Process</th>
<th>Outcome</th>
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<tr>
<td>( V(t) )</td>
<td>( v(t) )</td>
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<td>( V(j\omega), V(j\omega) )</td>
<td>( v(j\omega), v(j\omega) )</td>
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When context makes it clear whether \( \nu = \nu(t) \) or \( \nu = \nu(j\omega) \), we can simply write \( \nu \).

For stationary ergodic processes

\[
S_{\nu\nu}(j\omega) = S_{\nu\nu}(j\omega) = v(j\omega)v^*(j\omega) \quad \text{and} \quad S_{xy}(j\omega) = S_{xy}(j\omega) = x(j\omega)y^*(j\omega)
\]
Two-Port Noise Description

As we have seen in the prior lectures, through the method of circuit analysis, the internal noise generator of a circuit can be summed and represented by two noise generators $E_n$ and $I_n$.

The spectral densities of $E_n$ and $I_n$ must be calculated and specified. The cross spectral density must also be calculated and specified.
Calculating Total Noise

If the generator just has thermal noise, 
\[ \tilde{S}_{E_{N,\text{gen}}} = 4kTR_{\text{gen}} \]

Represent the combination of amplifier voltage and current noise by a single source 
\[ E_{\text{Total}} = E_N + I_N \cdot Z_{\text{gen}} \]

We can now calculate the spectral density of this total noise:
\[
\tilde{S}_{E_{\text{total,amplifier}}} = \tilde{S}_{E_n} + \| Z_g \|^2 \tilde{S}_{I_n} + 2 \text{Re}\left\{ \tilde{S}_{E_nI_n} Z^*_g \right\}
\]
\[
= \tilde{S}_{E_n} + \| Z_g \|^2 \tilde{S}_{I_n} + 2 \text{Re}\left\{ \tilde{S}_{E_nI_n} (R_{\text{gen}} - jX_{\text{gen}}) \right\}
\]
Signal / Noise Ratio of Generator

$V_{signal}, E_{n,total}$ and $E_{n,gen}$ are in series and see the same load impedance. The ratios of powers delivered by these will not depend upon the load. Therefore consider the available noise powers.

The signal power available from the generator is

$$P_{signal,available} = \frac{V_{signal,RMS}^2}{4R_{gen}}$$

If we consider a narrow bandwidth between $(f_{signal} - \Delta f / 2)$ and $(f_{signal} + \Delta f / 2)$, then the available noise power from $E_{N,gen}$ is

$$P_{noise,available,generator} = E[E_{n,gen}^2] = \tilde{S}(jf) \cdot \Delta f / 4R_{gen}$$

The signal/noise ratio of the generator is then

$$\text{SNR} = \frac{P_{signal,available}}{P_{noise,available,generator}} = \frac{V_{signal,RMS}^2 / 4R_{gen}}{\tilde{S}_{E_{n,gen}}(jf) \cdot \Delta f / 4R_{gen}} = \frac{V_{signal,RMS}^2 / 4R_{gen}}{kT \cdot \Delta f}$$
Signal / Noise Ratio of Generator+Amplifier

Signal power available from the generator: \[ P_{\text{signal available}} = \frac{V_{\text{signal RMS}}^2}{4R_{\text{gen}}} \]

Noise power available from generator: \[ P_{\text{noise avail, gen}} = \frac{S_V \cdot \Delta f}{4R_{\text{gen}}} = kT \cdot \Delta f \]

Noise power available from amplifier: \[ P_{\text{noise avail, Amp}} = \frac{\tilde{S}_{E_{\text{total, amplifier}}} \cdot \Delta f}{4R_{\text{gen}}} \]

Signal/noise ratio including amplifier noise:

\[ SNR = \frac{\frac{P_{\text{signal available}}}{P_{\text{noise avail, gen}} + P_{\text{noise avail, amp}}}}{\frac{V_{\text{signal RMS}}^2}{4R_{\text{gen}}} + \frac{\tilde{S}_{E_{\text{total}}} \cdot \Delta f}{4R_{\text{gen}}} + \frac{\tilde{S}_{E_{\text{gen}}} \cdot \Delta f}{4R_{\text{gen}}}} \]

\[ = \frac{\frac{V_{\text{signal RMS}}^2}{4R_{\text{gen}}}}{\frac{\tilde{S}_{E_{\text{total}}} \cdot \Delta f}{4R_{\text{gen}}} + kT \cdot \Delta f} \]
Noise Figure: Signal / Noise Ratio Degradation

Noise figure = \frac{\text{signal/noi se ratio before adding amplifier}}{\text{signal/noi se ratio before adding amplifier}}

\text{Signal/noi se ratio before adding amplifier} : \quad SNR = \frac{V_{signal RMS}^2}{kT \cdot \Delta f}

\text{Signal/noi se ratio after adding amplifier} : \quad SNR = \frac{V_{signal RMS}^2}{\tilde{S}_{E_{total}} \cdot \Delta f / 4R_{gen} + kT \cdot \Delta f}

\text{Noise figure} = F = \frac{\tilde{S}_{E_{total}} \cdot \Delta f / 4R_{gen} + kT \cdot \Delta f}{kT \cdot \Delta f}

\text{Noise figure} = 1 + \frac{\tilde{S}_{E_{total}} / 4R_{gen}}{kT} = 1 + \frac{\text{amplifier available input noise power}}{kT}
Calculating Noise Figure

Noise figure = \[ 1 + \frac{\widetilde{S}_{E_{\text{total}}}}{\frac{4}{\pi} R_{\text{gen}}} \]

We also know that:

\[ \widetilde{S}_{E_{\text{total}}} = \widetilde{S}_{E_{n}} + \| Z_{g} \|^2 \widetilde{S}_{I_{n}} + 2 \text{Re}\{\widetilde{S}_{E_{n}I_{n}} Z_{g}\} \]

We can calculate from this an expression for noise figure:

\[ F = 1 + \frac{\widetilde{S}_{E_{n}} + |Z_{s}|^2 \widetilde{S}_{I_{n}} + 2 \cdot \text{Re}(Z_{s}^{*} \widetilde{S}_{E_{n}I_{n}})}{4kTR_{\text{gen}}} \]
Minimum Noise Figure

Noise figure varies as a function of $Z_{\text{gen}} = R_{\text{gen}} + jX_{\text{gen}}$:

$$F = 1 + \frac{\tilde{S}_E + |Z_s|^2 \tilde{S}_{I_n} + 2 \cdot \text{Re}(Z^* \tilde{S}_{E_n I_n})}{4kTR_{\text{gen}}}$$

After some calculus, we can find a minimum noise figure and a generator impedance which gives us this minimum:

$$F_{\text{min}} = 1 + \frac{1}{4kT} \left[ 2 \sqrt{\frac{\tilde{S}_{E_n E_n}}{\tilde{S}_{I_n I_n}}} - \left( \text{Im} \left( \frac{\tilde{S}_{E_n I_n}}{\tilde{S}_{I_n I_n}} \right) \right)^2 + 2 \text{Re} \left( \frac{\tilde{S}_{E_n I_n}}{\tilde{S}_{I_n I_n}} \right) \right]$$

$$Z_{opt} = R_{opt} + jX_{opt} = \sqrt{\frac{\tilde{S}_{E_n E_n}}{\tilde{S}_{I_n I_n}}} - \left( \frac{\text{Im} \left( \frac{\tilde{S}_{E_n I_n}}{\tilde{S}_{I_n I_n}} \right) \right)^2 - j \cdot \frac{\text{Im} \left( \frac{\tilde{S}_{E_n I_n}}{\tilde{S}_{I_n I_n}} \right)}{\tilde{S}_{I_n I_n}}$$

Points to remember: (a) $F$ varies with $Z_{\text{gen}}$, (b) hence there is an optimum $Z_{\text{gen}}$ which gives a minimum $F(c)$. 
Noise Figure in Wave Notation

Written instead in terms of wave parameters:

\[ F = F_{\text{min}} + \frac{4r_n \cdot \| \Gamma_s - \Gamma_{\text{opt}} \|^2}{(1 - \| \Gamma_s \|^2)^2 \cdot (1 - \Gamma_{\text{opt}})^2} \]

These describe contours in the \( \Gamma_s - \Gamma \) plane of constant noise figure: "noise figure circles", i.e. a description of the variation of noise figure with source reflection coefficient \( t \).

I am not persuaded that there is value in here repeating the derivation of this expression.
Low-Noise Amplifier Design

Design steps are
1) in-band stabilization: this is best done at output port to avoid degrading noise
2) input tuning for $F_{\text{min}}$
3) output tuning (match)
4) out-of-band stabilization

Note that tuning for minimum noise figure requires a *mismatch* on the amplifier input; amplifier gain therefore must lie below the transistor MAG/MSG.

Note that tuning for minimum noise figure implies that amplifier input is mismatched: input reflection coefficient is therefore not zero!

Discrepancy in input noise-match & gain-match can be reduced by adding source inductance
Example LNA Design: 60 GHz, 130 nm SiGe BJT

gain & noise circles after input matching

note compromise between gain & noise tuning
Appendix
Appendix: Derivation of $F_{\text{min}}$ and $Z_{\text{opt}}$ (1)

$$4kT(F - 1) = \frac{\tilde{S}_{E_n}}{R_g} + \frac{(R_g^2 + X_g^2)\tilde{S}_{I_n}}{R_g} + 2 \cdot \text{Re}\left((R_g - jX_g)\tilde{S}_{E_nI_n}\right)$$

$$4kT(F - 1)R_g = \tilde{S}_{E_n} + (R_g^2 + X_g^2)\tilde{S}_{I_n} + 2 \cdot \text{Re}\left((R_g - jX_g)(\text{Re}(\tilde{S}_{E_nI_n}) + j \text{Im}(\tilde{S}_{E_nI_n}))\right)$$

$$4kT(F - 1)R_g = \tilde{S}_{E_n} + (R_g^2 + X_g^2)\tilde{S}_{I_n} + 2 \cdot \left(R_g \cdot \text{Re}(\tilde{S}_{E_nI_n}) + X_g \cdot \text{Im}(\tilde{S}_{E_nI_n})\right)$$

$$G = 4kT(F - 1) = \frac{\tilde{S}_{E_n}}{R_g} + R_g \tilde{S}_{I_n} + X_g^2 \tilde{S}_{I_n} / R_g + 2 \cdot \text{Re}(\tilde{S}_{E_nI_n}) + 2X_g \cdot \text{Im}(\tilde{S}_{E_nI_n}) / R_g$$

$$G = \frac{\tilde{S}_{E_n} + X_g^2 \tilde{S}_{I_n} + 2X_g \cdot \text{Im}(\tilde{S}_{E_nI_n})}{R_g} + R_g \tilde{S}_{I_n} + 2 \cdot \text{Re}(\tilde{S}_{E_nI_n})$$

$$\frac{dG}{dX} = \frac{2X_g \tilde{S}_{I_n} + 2 \cdot \text{Im}(\tilde{S}_{E_nI_n})}{R_g} = 0$$

$$X_{g, \text{opt}} = -\frac{\text{Im}(\tilde{S}_{E_nI_n})}{\tilde{S}_{I_n}} \text{ optimum generator reactance}$$
Appendix: Derivation of $F_{\text{min}}$ and $Z_{\text{opt}}$ (2)

$$G = 4kT(F - 1) = \frac{\tilde{S}_{E_n} + X_g^2 \tilde{S}_{I_n} + 2X_g \cdot \text{Im}(\tilde{S}_{E_n I_n})}{R_g} + R_g \tilde{S}_{I_n} + 2 \cdot \text{Re}(\tilde{S}_{E_n I_n})$$

Now substitute in the expression we have found for optimum generator reactance.

$$G = \frac{\tilde{S}_{E_n} + \left( \text{Im}(\tilde{S}_{E_n I_n}) / \tilde{S}_{I_n} \right)^2 \tilde{S}_{I_n} - 2 \cdot \left( \text{Im}(\tilde{S}_{E_n I_n}) \right)^2 / \tilde{S}_{I_n}}{R_g} + R_g \tilde{S}_{I_n} + 2 \cdot \text{Re}(\tilde{S}_{E_n I_n})$$

$$G = \frac{\tilde{S}_{E_n} - \left( \text{Im}(\tilde{S}_{E_n I_n}) \right)^2 / \tilde{S}_{I_n}}{R_g} + R_g \tilde{S}_{I_n} + 2 \cdot \text{Re}(\tilde{S}_{E_n I_n})$$

$$\frac{dG}{dR} = -\left( \frac{\tilde{S}_{E_n} - \left( \text{Im}(\tilde{S}_{E_n I_n}) \right)^2 / \tilde{S}_{I_n}}{R_g^2} \right) + \tilde{S}_{I_n} = 0$$

$$R_{g,\text{opt}} = \sqrt{\frac{\tilde{S}_{E_n} - \left( \text{Im}(\tilde{S}_{E_n I_n}) \right)^2 / \tilde{S}_{I_n}}{\tilde{S}_{I_n}}} \text{ optimum generator resistance}$$
Appendix: Derivation of $F_{\text{min}}$ and $Z_{\text{opt}}$ \textcolor{red}{(2)}

Now substitute in the expression for optimum generator resistance.

\begin{align*}
G_{\text{min}} &= \left( \tilde{S}_{E_n} - \left( \text{Im}(\tilde{S}_{E_nI_n}) \right)^2 / \tilde{S}_{I_n} \right) \left[ \tilde{S}_{I_n} + \tilde{S}_{I_n} \sqrt{\tilde{S}_{E_n} - \left( \text{Im}(\tilde{S}_{E_nI_n}) \right)^2 / \tilde{S}_{I_n}} \right] + 2 \cdot \text{Re}(\tilde{S}_{E_nI_n}) \\
G_{\text{min}} &= 4kT(F_{\text{min}} - 1) = 2 \left( \tilde{S}_{E_n} \tilde{S}_{I_n} \right) - \left( \text{Im}(\tilde{S}_{E_nI_n}) \right)^2 
+ 2 \cdot \text{Re}(\tilde{S}_{E_nI_n}) \\
F_{\text{min}} &= 1 + \frac{\left( \tilde{S}_{E_n} \tilde{S}_{I_n} - \left( \text{Im}(\tilde{S}_{E_nI_n}) \right)^2 \right)^{1/2} + \text{Re}(\tilde{S}_{E_nI_n})}{2kT} \text{ minimum noise figure!}
\end{align*}