ECE594I Notes set 11: Methods for Circuit Noise Analysis

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References and Citations:

Sources / Citations:
Kittel and Kroemer : Thermal Physics
Van der Ziel : Noise in Solid - State Devices
Papoulis : Probability and Random Variables (hard, comprehensive)
Wozencraft & Jacobs : Principles of Communications Engineering
Motchenbaker : Low Noise Electronic Design
Information theory lecture notes : Thomas Cover, Stanford, circa 1982
Probability lecture notes : Martin Hellman, Stanford, circa 1982
National Semiconductor Linear Applications Notes : Noise in circuits.

Suggested references for study.
Van der Ziel, Wozencraft & Jacobs, Peebles, Kittel and Kroemer
Papers by Fukui (device noise), Smith & Personik (optical receiver design)
National Semi. App. Notes (!)
Cover and Williams : Elements of Information Theory
summary
Goal: Computing Signal/Noise Ratio and Sensitivity

Radio Receiver

Optical Receiver

To compute the receiver sensitivity, we must find the signal/noise ratio at the decision circuit input.

It is often convenient to compare the input signal magnitude to the equivalent input-referred noise.
Goal: Computing Signal/Noise Ratio and Sensitivity

Each functional block of the radio receiver is a subcircuit
Each sub-circuit contains active and passive devices, all having noise models
There are a large # of noise generators within each circuit block
Goal: Computing Signal/Noise Ratio and Sensitivity

Earlier we found device terminal noise arising from internal fluctuations. Next we learn to compute the equivalent input noise of each sub-circuit:

From this we will find the total receiver input-referred noise:

Receiver SNR and sensitivity can then be found.
device noise models
(collecting results from prior lectures)
Thermal Noise

\[ \tilde{S}_{E_nE_n}(j\omega) = 4R \left( \frac{hf}{2} + \frac{hf}{\exp(hf/kT) - 1} \right) \]

\[ \tilde{S}_{I_nI_n}(j\omega) = \frac{4}{R} \left( \frac{hf}{2} + \frac{hf}{\exp(hf/kT) - 1} \right) \]

For \( hf \ll kT \) these become

\[ \tilde{S}_{E_nE_n}(j\omega) = 4kTR \]

\[ \tilde{S}_{I_nI_n}(j\omega) = \frac{4kT}{R} \]
For any component or complex network under thermal equilibrium (no energy supply)

\[ \frac{dP_{\text{available noise}}}{df} = kT \]

\[ \Rightarrow \tilde{S}_{E_nE_n}(jf) = 4kT \text{Re}(Z) \quad \text{or} \quad \tilde{S}_{I_nI_n}(jf) = 4kT \text{Re}(Y) \]

This follows from the 2\textsuperscript{nd} law of thermodynamics.
This allows quick noise calculation of complex passive networks
This allows quick noise calculation of antennas.

Biased semiconductor devices are NOT in thermal equilibrium.
Noise from an Antenna

\[ \frac{dP_{\text{available noise}}}{df} = kT \Rightarrow \tilde{S}_{E_nE_n}(jf) = 4kT \text{Re}(Z) \]

The antenna has both Ohmic and radiation resistances.

The Ohmic resistance has a noise voltage of spectral density \( 4kT_{\text{ambient}}R_{\text{Ohmic}} \), where \( T_{\text{ambient}} \) is the physical antenna temperature.

By the 2nd law, the radiation resistance has a noise voltage of spectral density \( 4kT_{\text{field}}R_{\text{rad}} \), where \( T_{\text{field}} \) is the average temperature of the region from which the antenna receives signal power.

Inter-galactic space is at 3.8 Kelvin....
Noise in PN and Schottky junctions

The diode current is

\[ I_{\text{diode}} = I_s \left( e^{qV/kT} - 1 \right) = I_s e^{qV/kT} - I_s = I_{\text{forward}} + I_{\text{reverse}} \]

Both the forward and reverse currents have shot noise, hence

\[ \tilde{S}_{I_{\text{diode}}} = 2qI_{\text{forward}} + 2qI_{\text{reverse}} = 2q(I_{\text{diode}} + 2I_s) \]

Under strong forward bias, \( \tilde{S}_{I_{\text{diode}}} = 2qI_{\text{diode}} \)

Under strong reverse bias, \( \tilde{S}_{I_{\text{diode}}} = 2qI_s \)

Under zero bias, \( \tilde{S}_{I_{\text{diode}}} = 4kT / r_{\text{diode}} \), as required by the 2nd law.
Shot noise and PN junctions: another model

For a strongly forward biased junction

\[ \tilde{S}_{I_{\text{diode}}} = 2qI_{\text{diode}} = 2kT / r_{\text{diode}} \text{ where } r_{\text{diode}} = kT / qI_{\text{diode}} \]

or

\[ \tilde{S}_{V_{\text{diode}}} = 2kTr_{\text{diode}} \]

hence

\[ \frac{dP_{\text{diode}}}{df} = kT / 2 \]

A biased diode has noise 1/2 that of a resistor of equal small-signal impedance. The factor of 2 arises from one-way current flow.
Bipolar Transistor Model---with Noise
Bipolar Noise Model

Collector shot noise
\[ \tilde{S}_{l_{nc}} = 2qI_c = 2kT / r_e = 2kT g_m \]

Base shot noise
\[ \tilde{S}_{l_{nb}} = 2qI_b = 2kT / r_{be} \]

We have found a slight correlation of \( I_{nb} \) and \( I_{nc} \) (a cross-spectral density) when \( 2\pi f (\tau_b + \tau_c) \) approaches 1. We will ignore this small effect.

The physical resistors \( (R_{bb}, R_{ex}, R_c) \) have thermal noise of spectral density \( \tilde{S}_v = 4kTR \)

\( R_{be} \) and \( r_e = 1 / g_m \) are not physical resistors.
The noise of \( R_{be} \) and \( r_e \) are the base and the collector shot noise generators.
FET Noise Model

\[ E_{N,Rg}, R_g, E_{N,Ri}, I_{Ng}, E_{N,Rd}, R_d, C_{gd}, I_{Nd}, E_{N,Rs}, R_s, V_{gs}, g_m V_{gs} e^{-j\omega \tau}, R_i, R_{ds}, C_{gs}, S \]
FET Noise model

\[ R_g, R_s, R_d \text{ are physical resistances } \Rightarrow d\langle V^2 \rangle/df = 4kTR \]

\[ I_{nd} \text{ is the thermal noise of the channel current} \]
\[ \tilde{S}_{I_{nd}} = 4kT \Gamma g_m \]
\[ \Gamma = 2/3: \text{ gradual - channel FET under constant mobility} \]
\[ \Gamma \sim 1-1.5: \text{ highly scaled FET under high - field conditions} \]

\[ R_i \text{ arises from the channel: } E_{N,Ri} \text{ and } I_{nd} \text{ have small correlation} \]
We will in further work ignore this small correlation

\[ \tilde{S}_{E_{N,Ri}} \approx 4kTR_i \]

\[ I_{ng} \text{ is the shot noise of the gate leakage current: } \tilde{S}_{I_{ng}} = 2qI_{gate} \]

\[ R_{ds} \text{ is not a physical resistor - no associated noise generator.} \]
Shot Noise of Heavily Attenuated Light

If the received power is much smaller than the transmitted power (losses are high) then the received power statistics will be dominated by the statistics of photon loss (Bernoulli trials).

With a mean received optical power $P_{out}$, the received power has fluctuations with spectral density

$$\tilde{S}_{P_{out}P_{out}} = 2h\nu P_{out}$$

This produces on a photodetector with quantum efficiency $\eta$ a photocurrent $I_{ph} = (\eta q / h\nu)P_{out}$ with a spectral density

$$\tilde{S}_{I_{out}I_{out}} = 2qI_{out}$$

If attenuation between source and detector is not small, it is easy to construct cases where $\tilde{S}_{P_{out}P_{out}} \neq 2h\nu P_{out}$
circuit noise calculations
Circuit noise analysis: Goals

The circuit output has both signal and noise.

\[ V_{out} = A_v V_{in} + V_{\text{noise, output}} \]

Noise arises from the generator, the amplifier, and the load

\[ V_{\text{noise, output}} = V_{\text{noise, generator}} + V_{\text{noise, amplifier}} + V_{\text{noise, load}} \]

These noise terms can be represented by fictitious input terms:

\[ V_{out} = A_v V_{in} + V_{\text{noise, output}} = A_v \left( V_{in} + V_{\text{noise, input}} \right), \text{ where } V_{\text{noise, input}} = V_{\text{noise, output}} / A_v \]

How do we calculate the output-referred noise?
Noise model of this circuit

The circuit has a large number of noise generators. Noise analysis of most practical circuits is of formidable complexity. Brute-force methods are too hard for hand analysis. We will learn more efficient techniques. We will illustrate calculations with a very simple circuit.
Simple amplifier, with simplified noise model.

Notation: Single - sided, Hz - based spectral densities

\[
\tilde{S}_{V_nV_n} = 4kTR \quad \text{or} \quad \tilde{S}_{I_nI_n} = 4kTG \quad \text{for all resistors}
\]

\[
\tilde{S}_{I_{nd}I_{nd}} = 4kTG_m \quad \text{for the FET channel noise.}
\]
Now calculate the output voltage:

\[
V_{out} = \left( V_{gen} + E_{N,R_{gen}} + E_{N,R_g} \right) \left( 1 + j2\pi f C_{gs} \left( R_g + R_{gen} \right) \right)^{-1} \left( - g_m R_L \right) \\
+ \left( I_{N,d} + I_{N,R_L} \right) R_L \\
= V_{otsignal} + V_{out,amp\_noise} + V_{outgen\_noise}
\]

\[
V_{otsignal} = V_{gen} \left( 1 + j2\pi f C_{gs} \left( R_g + R_{gen} \right) \right)^{-1} \left( - g_m R_L \right)
\]

\[
V_{out,amp\_noise} = E_{N,R_g} \left( 1 + j2\pi f C_{gs} \left( R_g + R_{gen} \right) \right)^{-1} \left( - g_m R_L \right) + \left( I_{N,d} + I_{N,R_L} \right) R_L
\]

\[
V_{outgen\_noise} = E_{N,R_{gen}} \left( 1 + j2\pi f C_{gs} \left( R_g + R_{gen} \right) \right)^{-1} \left( - g_m R_L \right)
\]

where

\[
\tilde{S}_{V_nV_n}(jf) = 4kTR \text{ or } \tilde{S}_{I_nI_n}(jf) = 4kTG \text{ for all resistors}
\]

\[
\tilde{S}_{I_{nd}I_{nd}}(jf) = 4kTg_m \text{ for the FET channel noise.}
\]
Reminder

If
\[ V_y(jf) = H(jf)V_x(jf) \]

Then
\[
\tilde{S}_{V_yV_x}(jf) = H(jf)\tilde{S}_{V_yV_x}(jf) \\
\tilde{S}_{V_xV_y}(jf) = \tilde{S}_{V_xV_y}(jf)H^*(jf) \\
\text{and} \\
\tilde{S}_{V_yV_y}(jf) = \|H(jf)\|^2 \tilde{S}_{V_xV_x}(jf)
\]
So:

\[ V_{\text{out signal}} = V_{\text{gen}} \left(1 + j2\pi f C_{gs} (R_g + R_{gen})\right)^{-1}(-g_m R_L) \]

\[ \tilde{S}_{V_{\text{amp, out}}} (jf) = \tilde{S}_{N,R_g} \frac{\left(g_m R_L\right)^2}{1 + \left(2\pi f C_{gs}\right)^2 (R_g + R_{gen})^2} + \left(\tilde{S}_{I_{N,d}} + \tilde{S}_{I_{N,RL}}\right) (R_L)^2 \]

\[ \tilde{S}_{V_{\text{gen, out}}} (jf) = \tilde{S}_{N,R_{gen}} \frac{\left(g_m R_L\right)^2}{1 + \left(2\pi f C_{gs}\right)^2 (R_g + R_{gen})^2} \]

where

\[ \tilde{S}_{V_{nV_n}} (jf) = 4kTR \quad \text{or} \quad \tilde{S}_{I_{nI_n}} (jf) = 4kTG \quad \text{for all resistors} \]

\[ \tilde{S}_{I_{ndI_{nd}}} (jf) = 4kT \Gamma g_m \quad \text{for the FET channel noise.} \]
The output signal

\[ V_{\text{outsignal}} = V_{\text{gen}} \left( 1 + j 2 \pi f C_{gs} \left( R_g + R_{\text{gen}} \right) \right)^{-1} \left( - g_m R_L \right) \]

\[ = A_v \left( j 2 \pi f \right) V_{\text{gen}} \]

The output noise *due to the amplifier*

\[ \tilde{S}_{V_{\text{amp,out}}} \left( j f \right) = 4kT R_g \frac{\left( g_m R_L \right)^2}{1 + \left( 2 \pi f C_{gs} \right)^2 \left( R_g + R_{\text{gen}} \right)^2} + \left( 4kT g_m + \frac{4kT}{R_L} \right) \left( R_L \right)^2 \]

The output noise *due to the generator*

\[ \tilde{S}_{V_{\text{gen,out}}} \left( j f \right) = 4kT R_{\text{gen}} \frac{\left( g_m R_L \right)^2}{1 + \left( 2 \pi f C_{gs} \right)^2 \left( R_g + R_{\text{gen}} \right)^2} \]
Now Define *equivalent input noise*

\[
V_{out} = A_v(\jmath f) \cdot V_{gen} + V_{out,amp\_noise} + V_{out,gen\_noise}
\]

\[
= A_v(\jmath f) \cdot (V_{gen} + V_{in,amp\_noise} + V_{in,gen\_noise})
\]

This means simply: \(V_{in,gen\_noise} = E_{N,gen}\) and \(V_{in,amp\_noise} = V_{out,amp\_noise} / A_v(\jmath f)\)

So the amplifier input - referred noise is:

\[
\tilde{S}_{\text{amp,in}}(\jmath f) = \frac{\tilde{S}_{\text{amp,out}}(\jmath f)}{\| A_v(\jmath f) \|^2} = \tilde{S}_{\text{amp,out}}(\jmath f) \cdot \frac{1 + (2\pi f C_{gs})^2 (R_g + R_{gen})^2}{(g_m R_L)^2}
\]

And the input noise *due to the generator* *is*:

\[
\tilde{S}_{V_{gen,in}}(\jmath f) = 4kT R_{gen}
\]
\[ \tilde{S}_{V_{\text{amp,in}}} (jf) = 4kTR_g \text{ input - referred noise from } R_g \]

\[ + 4kT \Gamma g_m \cdot \left( \frac{1}{g_m} \right)^2 \left( 1 + \left( 2\pi f C_{gs} \right)^2 \left( R_g + R_{\text{gen}} \right)^2 \right) \]

input referred channel noise

\[ + \left( \frac{4kT}{R_L} \right) \left( \frac{1}{g_m^2} \right)^2 \left( 1 + \left( 2\pi f C_{gs} \right)^2 \left( R_g + R_{\text{gen}} \right)^2 \right) \]

input - referred load resistor noise

Input noise * from the generator *

\[ \tilde{S}_{V_{\text{gen,in}}} (jf) = 4kTR_{\text{gen}} \]
Circuit Noise Analysis: 1st Example (g)

Noise Figure definition:

\[
F = \frac{\tilde{S}_{V_{gen,in}}(jf) + \tilde{S}_{V_{amp,in}}(jf)}{\tilde{S}_{V_{gen,in}}(jf)}
\]

From which we can write

\[
\tilde{S}_{V_{in,total\_noise}}(jf) = 4kTR_{gen}F
\]

Signal/Noise ratio:

\[
\text{SNR} = \frac{\tilde{S}_{V_{signal}}(jf)}{\tilde{S}_{V_{in,total\_noise}}(jf)}
\]

Where \(\tilde{S}_{V_{signal}}(jf)\) is the input signal's power spectral density.
These are the exact steps for calculation of input-referred noise, output-referred noise, SNR, and noise figure.

This is how a computer might calculate these.

The method is extremely tedious, even for a small circuit.

Note that, in computing input-referred noise, many of our calculation steps were cancelled one we found the final answer.

Clearly, then, our method must be inefficient.
Circuit Noise Analysis: 1st Example: Summary

Analysis was hard because we
.....propagated the circuit noise generators to the circuit output,
...then propagated them back to the input.

This involves cancelled steps--extra work.

It is particularly inefficient because
***The most important noise sources are near the input***
Circuit Noise Analysis: Source Transposition Method

Let use move all the circuit noise generators to the circuit input.
We must restrict ourselves to transformations which do not change the 2-port input-output characteristics of the network between input and output.

This means, make transformations only inside red box
Circuit Noise Analysis: Source Transposition Method

Thevenin - Norton

\[ E_N = I_N R \]

Moving Current Across A Branch

Output - Input

\[ I_{out} = g_m (V^+ - V^-) \quad E_N = I_N / g_m \]

Moving Voltage Through A Node

\[ E_N / A_v \]
"Walk" $I_{NRL}$ to the input

\[ R_{gen} \quad G \quad R_g \quad C_{gs} \quad g_m V_{gs} \quad D \quad I_{NRL} \quad R_L \quad V_{out} \]

\[ V_{gen} \quad V_{in} \quad V_{gs} \quad S \]

\[ I_{NRL}/g_m \]

\[ C_{gs} \]

\[ V_{out} \]

\[ V_{in} \]
Circuit Noise Analysis: Source Transposition Method

\[ V_{\text{gen}} + \]
\[ r_{\text{gen}} \]
\[ I_{\text{NRL}} \times \frac{1}{g_m} \left( 1 + j2\pi f \frac{C_{gs}}{R_g} \right) \]
\[ I_{\text{NRL}} \times j2\pi f \frac{C_{gs}}{g_m} \]

\[ V_{\text{gen}} - \]
\[ r_{\text{gen}} \]
\[ I_{\text{NRL}} \times \frac{1}{g_m} \left( 1 + j2\pi f \frac{C_{gs}}{R_g} \right) \]
\[ I_{\text{NRL}} \times j2\pi f \frac{C_{gs}}{g_m} \]

\[ I_{\text{NRL}} \times \left[ \frac{1}{g_m} \left( 1 + j2\pi f \frac{C_{gs}}{R_g} \right) + \frac{1}{g_m} j2\pi f \frac{C_{gs}}{R_{\text{gen}}} \right] \]
\[ = I_{\text{NRL}} \times \left[ \frac{1}{g_m} + j2\pi f \frac{C_{gs}}{R_{g} + R_{\text{gen}}} \right] / g_m ] \]
Circuit Noise Analysis: Source Transposition Method

\[ \tilde{S}_{V_{\text{input};R_{L\text{noise}}}}(jf) = \left( \frac{4kT}{R_L} \right) \left( \frac{1}{g_m^2} + \frac{\left(2\pi f C_{gs}\right)^2 \left(R_g + R_{gen}\right)^2}{g_m^2} \right) \]

This was certainly not an easy calculation, but because \( R_L \) is far from the input, it was the single hardest calculation to make.

The channel noise current generator is in parallel with that of \( R_L \) so,

\[ \tilde{S}_{V_{\text{input};\text{channel noise}}}(jf) = (4kT \Gamma g_m) \left( \frac{1}{g_m^2} + \frac{\left(2\pi f C_{gs}\right)^2 \left(R_g + R_{gen}\right)^2}{g_m^2} \right) \]
In this particularly easy example, we can also see that

\[ \tilde{S}_{V_{\text{input}}; R_g} (jf) = 4kT R_g \quad \tilde{S}_{V_{\text{input}}; \text{generator}} (jf) = 4kT R_{\text{gen}} \]

so

\[ \tilde{S}_{V_{\text{input}}; \text{amplifier}} (jf) = 4kT R_g + \left( 4kT \frac{g_m}{R_L} + \frac{4kT}{R_L} \right) \left( \frac{1}{g_m^2} + \frac{(2\pi f C_{gs})^2 (R_g + R_{\text{gen}})^2}{g_m^2 R_L} \right) \]

hence

\[ F = 1 + \frac{\tilde{S}_{V_{\text{input}}; \text{amplifier}}}{\tilde{S}_{V_{\text{input}}; \text{generator}}} = 1 + \frac{R_g}{R_{\text{gen}}} + \frac{1}{R_{\text{gen}}} \left( \frac{4kT}{g_m} + \frac{4kT}{g_m R_L} \right) \left( 1 + (2\pi f C_{gs})^2 (R_g + R_{\text{gen}})^2 \right) \]
We frequently wish to specify noise of a device or circuit with the generator impedance unknown and unspecified.

The $E_n - I_n$ representation allows this.
En-In Model: Source Transposition Again

Once again, "walk" sources to input - - but not into the generator illustration of load resistor noise only.
En-In Model: Source Transposition Again

The output noise is represented at $V_{in}$ by a combination of a voltage source and a current source.

As they are both related 1:1 to $I_{NRL}$, they are 100% correlated.
En-In Model: With All Sources

Because

\[ E_{n,\text{total}} = I_{NRL} \left( \frac{1}{g_m} \right) \left( 1 + j2\pi f C_{gs} R_g \right) + I_{nd} \left( \frac{1}{g_m} \right) \left( 1 + j2\pi f C_{gs} R_g \right) + E_{NRG} \]

\[ I_{n,\text{total}} = I_{NRL} \left( j2\pi f C_{gs} / g_m \right) + I_{nd} \left( j2\pi f C_{gs} / g_m \right) \]

And Because \( \tilde{S}_{E_{NRG}} (jf) = 4kT R_g \)  \( \tilde{S}_{I_{NRL}} (jf) = 4kT / R_L \)  \( \tilde{S}_{I_{nd}} (jf) = 4kT g_m \)

\[ \tilde{S}_{E_{n,\text{total}}} (jf) = \left( \frac{4kT}{R_L} + 4kT g_m \right) \left( \frac{1}{g_m} \right)^2 \left( 1 + \left( 2\pi f C_{gs} R_g \right)^2 \right) + 4kT R_g \]

\[ \tilde{S}_{I_{n,\text{total}}} (jf) = \left( \frac{4kT}{R_L} + 4kT g_m \right) \left( 2\pi f C_{gs} / g_m \right)^2 \]

\[ \tilde{S}_{E_{n,\text{total}}} (jf) = \left( \frac{4kT}{R_L} + 4kT g_m \right) \left( \frac{1}{g_m} \right) \left( 1 + j2\pi f C_{gs} R_g \right) \left( j2\pi f C_{gs} \right)^* \]

Note in particular the cross spectral density.
Consider what we have done:

An amplifier, with the generator not specified or present, is represented by a pair of correlated noise generators at its input.

Later, when a generator impedance is specified, $\tilde{S}_{E_n}$, $\tilde{S}_{I_n}$, and $\tilde{S}_{E_n I_n}$ can be used to calculate the total input noise (voltage, current, or available power).
Using the En-In Model to Compute total Noise

Given a circuit with specified \( \tilde{S}_{E_n,\text{total}}E_n,\text{total} \) (if), \( \tilde{S}_{I_n,\text{total}}I_n,\text{total} \) (if), and \( \tilde{S}_{E_n,\text{total}}I_n,\text{total} \) (if), and given a specified generator impedance \( Z_{\text{gen}} = R_{\text{gen}} + jX_{\text{gen}} \)

\[
E_{n,\text{total,amplifier}} = E_n + I_N Z_g
\]

So

\[
\tilde{S}_{E_n,\text{total,amplifier}} = \tilde{S}_{E_n} + \| Z_g \|^2 \tilde{S}_{I_n} + 2 \text{Re}\left\{ \tilde{S}_{E_n}I_n Z_g^* \right\}
\]

\[
= \tilde{S}_{E_n} + \| Z_g \|^2 \tilde{S}_{I_n} + 2 \text{Re}\left\{ \tilde{S}_{E_n}I_n \left( R_{\text{gen}} - jX_{\text{gen}} \right) \right\}
\]
Using the En-In Model--Conclusion

If we use the circuit relationship

$$\tilde{S}_{E_{n,total,amplifier}} = \tilde{S}_{E_n} + \|Z_g\|^2 \tilde{S}_{I_n} + 2 \text{Re}\{\tilde{S}_{E_nI_n} Z_g^*}\}$$

$$= \tilde{S}_{E_n} + \|Z_g\|^2 \tilde{S}_{I_n} + 2 \text{Re}\{\tilde{S}_{E_nI_n} (R_{gen} - jX_{gen})\}$$

and the device relationships

$$\tilde{S}_{E_{n,total}E_{n,total}} (jf) = \left(\frac{4kT}{R_L} + 4kT \Gamma g_m\right) \left(\frac{1}{g_m}\right)^2 \left(1 + \left(2\pi f C_{gs} R_g\right)^2\right) + 4kTR_g$$

$$\tilde{S}_{I_{n,total}I_{n,total}} (jf) = \left(\frac{4kT}{R_L} + 4kT \Gamma g_m\right) \left(2\pi f C_{gs} / g_m\right)^2$$

$$\tilde{S}_{E_{n,total}I_{n,total}} (jf) = \left(\frac{4kT}{R_L} + 4kT \Gamma g_m\right) \left(\frac{1}{g_m}\right) \left(1 + j2\pi f C_{gs} R_g\right) \left(j2\pi f C_{gs}\right)^*$$

Then by varying $Z_g$ (calculus), we can find the device minimum noise figure and the optimum source impedance which provides this, i.e. we can calculate the Fukui FET noise figure expression.
by picture

$\frac{g}{hV}\begin{array}{c}R_{in}\end{array}$

$V_{OL} = -g_{m} n R_{L} V_{gs}$

amplifier 1st stage

detailed model

amplifier 2nd stage

highly idealized

$A_{v} = A_{2} \cdot (g_{m} n R_{L})$
\[ I_1: \delta I_1 = \frac{4kT}{R_F} \]

\[ I_2: \delta I_2 = \frac{4kT}{R_L} \eta_m + \frac{4kT}{R_L} \]

apply transformations which do not change \( V_{in} \) or \( V_{out} \)
So the total input referred noise current spectral density is

\[ \frac{d}{df} \langle I_i^2 I_i^* \rangle = \frac{4kT}{R_f} + 4kT \Pi \frac{9\mu}{q_n} \left[ \frac{\omega^2 G^2}{q_n^2} + \frac{1}{q_n^2 R^2} \right] \]

\[ = \frac{4kT}{R_f} + 4kT \Pi \frac{9\mu}{q_n} \left[ \frac{\omega^2 G^2}{q_n^2} + \frac{1}{R_f^2} \right] \]

... the method is faster than the lecture showed, as we can move generator on 1 drawing...
Comment about biasing

Bias tees are noiseless:

\[ I_{dc} = \frac{V_{cc} - V_i}{R} = \frac{\Delta V}{R} \]

\[ \frac{1}{I_{noise \, in}} = 4kT/R = 4kT \frac{I_{dc}}{\Delta V} \]

Make \( \Delta V \) big to make the current source noiseless.
Fat Constant - current sense

at low frequencies:

\[
\frac{d \langle i_n \Delta i_n^* \rangle}{dt} = g_k T I \frac{q_n}{I_{dc}} \cdot I_{dc}
\]

Since \((g_n I_{dc})^{-1} \sim O(V_p)\)...

\[
\frac{d \langle i_n \Delta i_n^* \rangle}{dt} \sim O\left[4 k T I I_{dc} \frac{I_{dc}}{V_p}\right]
\]

...this is noiser than the resistor by the ratio \(n(\Delta V \cdot \frac{I_p}{V_p})\)

Ouch!
I_{ref} noiseless (?) \rightarrow (really \ show I_{ref}(0V))
\[ I_n = (E_{rd} + E_{rs}) \cdot \frac{\Gamma_n}{\Gamma_{n} + \Gamma_{dl} + \Gamma_{ds}} \cdot g_m \]
\[ + \left( \Gamma_{ds} + \Gamma_{dl} \right) \frac{\Gamma_n}{\Gamma_{n} + \Gamma_{dl} + \Gamma_{ds}} \cdot I_{in} \cdot g_m + I_{in} \]

Use \( \beta \cdot g_m = \Gamma_n \):

\[ A_{in\infty}(f) = 4kT \left( \frac{\Gamma_{ds} + \Gamma_{dl}}{2} \right) \left( \frac{\beta}{\Gamma_{ds} + \Gamma_{dl} + \beta \cdot g_m} \right)^2 \]
\[ + 2g_0 \left( \frac{\beta}{\Gamma_{ds} + \Gamma_{dl} + \beta \cdot g_m} \right)^2 \left( \Gamma_{ds} + \Gamma_{dl} \right)^2 \]
\[ + 2g_0 \cdot I_{dc} \]

This gets complicated. The last term alone (2g_0) is bigger than the resistor current-source noise by the ratio \( \frac{\Delta V/2}{kT\beta} \)

\[ - \Delta V \]

\[ \text{Ouch!} \]
Simple common-source amplifier:

$$V_{gs} \rightarrow \text{FET} \rightarrow R_L$$

Since this is just an example, pick a simplified FET model.

$$E_{ri}: \frac{V_{gs}}{g_{m}} \rightarrow \text{FET} \rightarrow \frac{V_{gs}}{g_{m}}$$

$$E_{in}: S = 4kTf_{in} \{ \text{zero cross-spectral density} \}$$

$$I_{in}: S = 4kTf_{m}g_{m}$$
\[ V_i : 4kT R_g \]
\[ V_2 : 4kT R_i \]
\[ V_3 : 4kT R_i \]

\[ I_i : 4kT 7q_n + 4kT \left( \frac{1}{R_i} + \frac{1}{R_L} \right) \quad \text{Reg} = R_d s + R_i + R_L \]

\[
\frac{V_{oc}}{V_{in}} = \frac{R_g}{R_g + R_s} \frac{1}{1 + j \omega C g_s \left( R_i + R_s || R_g \right)} \quad (-g_m \text{Reg})
\]

Most efficient to work this by a mixture of methods.

A V_i already at input: leave it there...

\[
\frac{V_{oc}}{I_i} = \text{Reg}
\]
So, we can model $I_1$ by an input voltage of

$$\frac{V_{eq1}}{I_1} = -I_1 \cdot \frac{R_g + R_s}{R_g} \left( 1 + j\omega C_g S (R_i + R_g) \right)$$

... easier to do than draw...
So we can now gather terms & write the input-referred noise voltage...
This is a total noise voltage model

\[ S_{\text{V_{\text{total}}} \text{(f)}} = 4kT E_{\text{g}} m \cdot \left\{ \frac{R_{g} + R_{s}}{9m R_{g}} \right\}^{2} \left[ 1 + C_{g}^{2} \cdot (R_{i} + R_{s}/R_{g})^{2} \right] \]

\[ + 4kT R_{s} \]

\[ + 4kT R_{i} \cdot \left[ 1 + R_{s}/R_{g} \right]^{2} \]

\[ + 4kT R_{g} \cdot \left[ R_{s}/R_{g} \right]^{2} \]

You may simplify further...

\[ \text{input} \]
\[ \text{referred total noise voltage} \]
\[ \text{spectral density, } \nu^{1/2} \]