References and Citations:
Sources / Citations :
Kittel and Kroemer : Thermal Physics
Van der Ziel : Noise in Solid - State Devices
Papoulis : Probability and Random Variables (hard, comprehensive)
Wozencraft & Jacobs : Principles of Communications Engineering.
Motchenbaker : Low Noise Electronic Design
Information theory lecture notes : Thomas Cover, Stanford, circa 1982
Probability lecture notes : Martin Hellman, Stanford, circa 1982
National Semiconductor Linear Applications Notes : Noise in circuits.

Suggested references for study.
Van der Ziel, Wozencraft & Jacobs, Peebles, Kittel and Kroemer
Papers by Fukui (device noise), Smith & Personik (optical receiver design)
National Semi. App. Notes (!)
Cover and Williams : Elements of Information Theory
random processes
Random Processes

Draw a set of graphs, on separate sheets of paper, of functions of voltage vs. time.

Put them into a garbage can.

This garbage can is called the probability sample space.

Pick out one sheet at random. This is our random function of time.

The random process is $V(t)$.
The particular outcome is $v(t)$.
Time Averages vs. Sample Space Averages

Recall the definition of the expectation of a function $g(X)$ of a random variable $X$

$$E[g(x)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx = \bar{g}$$

$\bar{g}$ is the *average value* of $g$, where the average is over the sample space.

With our random process definition, we can define an average over the sample space at some particular time $t_1$:

$$E[g(v(t_1))] = \int_{-\infty}^{+\infty} g(v(t_1)) f_V(v(t_1)) d(v(t_1))$$

We can also define an average of any one outcome function over time:

$$A[g(v_i(t))] = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} g(v_i(t)) dt$$
What is Random in a Random Process?

We somehow have the expectation that the signal "varies randomly with time".

Yet, in our definition, it is the selection of the sample which is random.

Ergodicity addresses this discrepancy.

Its implications are surprising.
Ergodic Random Processes

An Ergodic random process has averages over time equal to averages over the statistical sample space

\[ E[g(v(t_i))] = A[g(v_i(t))] \text{ for all } i, \text{ all } t_i \]

In some sense, we have made "random variation with time" equivalent to "random variation over the sample space"
With time samples at times $t_1$ and $t_2$ the random process $V(t)$ has values $V(t_1)$ and $V(t_2)$.

$V(t_1)$ and $V(t_2)$ have some joint probability distribution. They might (or might not) be jointly Gaussian.
Random Waveforms are Random Vectors

Using Nyquist's sampling theorem, if a random signal is bandlimited, and if we pick regularly-spaced time samples $t_1...t_n$, we convert our random process into a random vector.

We can thus analyze random signals using vector analysis and geometry.
Stationary Random Processes

The statistics of a stationary process do not vary with time.

\[ N^{th} \text{ - order stationarity:} \]
\[ E[f(V(t_1), V(t_2), \ldots, V(t_n))] = E[f(V(t_1 + \tau), V(t_2 + \tau), \ldots, V(t_n + \tau))]. \]

..and lower orders

\[ 2^{nd} \text{ - order stationarity:} \]
\[ E[f(V(t_1), V(t_2))] = E[f(V(t_1 + \tau), V(t_2 + \tau))]. \]

lower orders \[ \rightarrow E[f(V(t))] = E[f(V(t_1 + \tau))]. \]
Restrictions on the random processes we consider

We will make following restrictions to make analysis tractable:

The process will be **Ergodic**.

The process will be **stationary** to any order: all statistical properties are independent of time. Many common processes are not stationary, including integrated white noise and 1/f noise.

The process will be **Jointly Gaussian**. This means that if the values of a random process $X(t)$ are sampled at times $t_1$, $t_2$, etc, to form random variables $X_1=X(t_1)$, etc, then $X_1,X_2$, etc. are a jointly Gaussian random variable.

In nature, many random processes result from the sum of a vast number of small underlying random processes. From the central limit theorem, such processes can frequently be expected to be Jointly Gaussian.
Variation of a random process with time

For the random process $X(t)$, look at $X_1=X(t_1)$ and $X_2=X(t_2)$.

$$R_{X_1X_2} = E[X_1X_2] = \int \int x_1x_2 \cdot f_{X_1X_2}(x_1, x_2) dx_1 dx_2$$

To compute this we need to know the joint probability distribution. We have assumed a Gaussian process. The above is called the Autocorrelation function. IF the process is stationary, it is a function only of $(t_1-t_2)=\tau$, and hence

$$R_{XX}(\tau) = E[X(t)X(t + \tau)]$$

this is the autocorrelation function. It describes how rapidly a random voltage varies with time….

PLEASE recall we are assuming zero-mean random processes (DC bias subtracted). Thus the autocorrelation and the auto-covariance are the same
Variation of a random process with time

Note that \( R_{XX}(0) = E[X(t)X(t)] = \sigma_X^2 \) gives the variance of the random process.

The autocorrelation function gives us variance of the random process and the correlation between its values for two moments in time. If the process is Gaussian, this is enough to completely describe the process.

Narrow autocorrelation:
- Fast variation

Broad autocorrelation
- Slow variation
Autocorrelation is an Estimate of the Variation with Time

If random variables X and Y are Jointly Gaussian, and have zero mean, then knowledge of the value y of the outcome of Y results in a best estimate of X as follows:

\[ E[X|Y = y] = \langle X|Y = y \rangle = \frac{R_{XY}}{\sigma_Y^2} y \]

"The expected value of the random variable X, given that the random variable Y has value y is ..."

Hence, the autocorrelation function tells us the degree to which the signal at time \( t \) is related to the signal at time \( t + \tau \)

A narrow autocorrelation is indicative of a quickly-varying random process
Power spectral densities

The autocorrelation function describes how a random process evolves with time.

Find its Fourier transform:

\[ S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) \exp(-j\omega \tau) d\tau \]

This is called the power spectral density of the signal.

Remembering the usual Fourier transform relationships, if the power spectrum is broad, the autocorrelation function is narrow, and the signal varies rapidly--it has content at high frequencies, and the voltages of any two points are strongly related only if the two points are close together in time.

If the power spectrum is narrow, the autocorrelation function is broad, and the signal varies slowly--it has content only at low frequencies and the voltages of any two points are strongly related unless if the two points are broadly separated in time.
Power spectral densities

\[ R_{xx}(\tau) \times x(t) \]

\[ S_{xx}(\omega) \times x(t) \]
Power Spectral Densities

Recall that the power spectral density is the Fourier transform of the autocorrelation function

\[ S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) \exp(-j\omega\tau) d\tau \]

The inverse transform holds, so that

\[ R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) \exp(j\omega\tau) d\omega \]

Specifically,

\[ R_{XX}(0) = \sigma^2_x = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega \]

So, if \( \sigma^2_x \) is called the power in the process, then integrating the power spectral density will give us the power.

This is the justification for the term, "power spectral density"
Back To Ergodicity (1)

Statistical autocorrelation (average over sample space):
\[ R_{XX}(\tau) = E[X(t)X(t + \tau)]. \]

Time autocorrelation function (note subscripts) of a particular outcome \( x_i(t) \) of the random process \( X(t) \).
\[ R_{x_i x_i}(\tau) = A[x_i(t)x_i(t + \tau)] = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_i(t)x_i(t + \tau) dt \]

If \( X(t) \) is Ergodic, then \( R_{x_i x_i}(\tau) = R_{XX}(\tau) \) for *all* outcomes \( x_i(t) \).
If $X(t)$ is Ergodic, then $R_{x_i x_i}(\tau) = R_{XX}(\tau)$ for *all* outcomes.

$$R_{x_i x_i}(\tau) = A[x_i(t) x_i(t + \tau)] = R_{XX}(\tau)$$

But $R_{x_i x_i}(\tau) = \mathcal{F}^{-1}\{S_{x_i x_i}(j\omega)\} = \mathcal{F}^{-1}\{X_i(j\omega)X_i^*(j\omega)\}$

where (notation problems) $X_i(j\omega)$ is the Fourier transform of $x_i(t)$.

So, every outcome $x_i(t)$ has the same Fourier Magnitude $\|X(j\omega)\|$. Each outcome $x_i(t)$ likely has a different Fourier phase.
Correlated Random Processes

Two processes can be statistically related. Consider two random processes $X(t)$ and $Y(t)$.

Define the cross-correlation function of the processes

$$R_{XY}(\tau) = E[X(t)Y(t + \tau)]$$

They will have a cross-spectral density as follows:

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau) \exp(-j\omega\tau) d\tau$$

and therefore

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XY}(\omega) \exp(j\omega\tau) d\omega$$
Single-Sided Hz-based Spectral Densities

Double - Sided Spectral Densities

\[ R_{XX}(\tau) = E[X(t)X(t + \tau)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(j\omega) \exp(j\omega\tau) d\omega \]

\[ S_{XX}(j\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) \exp(-j\omega\tau) d\tau \]

Single - Sided Hz - based Spectral Densities

\[ R_{XX}(\tau) = E[X(t)X(t + \tau)] = \frac{1}{2} \int_{-\infty}^{+\infty} \tilde{S}_{XX}(jf) \exp(j2\pi f\tau) df \]

\[ \tilde{S}_{XX}(jf) = 2 \int_{-\infty}^{+\infty} R_{XX}(\tau) \exp(-j2\pi f\tau) d\tau \]
Single-Sided Hz-based Spectral Densities- Why ?

Why this notation?
The signal power in the bandwidth \( \{ f_{\text{low}}, f_{\text{high}} \} \)

\[
\text{Power} = \frac{1}{2} \int_{-f_{\text{high}}}^{-f_{\text{low}}} \tilde{S}_{XX}(jf) \, df + \frac{1}{2} \int_{f_{\text{low}}}^{f_{\text{high}}} \tilde{S}_{XX}(jf) \, df = \int_{f_{\text{low}}}^{f_{\text{high}}} \tilde{S}_{XX}(jf) \, df
\]

\( \rightarrow \tilde{S}_{XX}(jf) \) is directly the Watts of signal power per Hz of signal bandwidth at frequencies lying close to the frequency \( f \).
Double-Sided Cross Spectral Densities

\[ R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XY}(j\omega)\exp(j\omega\tau)d\omega \]

\[ S_{XY}(j\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau)\exp(-j\omega\tau)d\tau \]

Single-Sided Hz-based Cross Spectral Densities

\[ R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \frac{1}{2} \int_{-\infty}^{+\infty} \tilde{S}_{XY}(jf)\exp(j2\pi f\tau)df \]

\[ \tilde{S}_{XY}(jf) = 2 \int_{-\infty}^{+\infty} R_{XY}(\tau)\exp(-j2\pi f\tau)d\tau \]

\[ \tilde{S}_{XY}(jf) \text{ is also often written as } \frac{d}{df} \langle XY \rangle \]
Example: Cross Spectral Densities

\[ V(t) = X(t) + Y(t) \]

\[ R_{vv}(\tau) = E\left[ (X(t) + Y(t))(X(t + \tau) + Y(t + \tau)) \right] = R_{xx}(\tau) + R_{yy}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) \]

\[ S_{vv}(j\omega) = S_{xx}(j\omega) + S_{yy}(j\omega) + S_{xy}(j\omega) + S_{yx}^*(j\omega) = S_{xx}(j\omega) + S_{yy}(j\omega) + 2 \cdot \text{Re}\{S_{xy}(j\omega)\} \]

Or, in single-sided spectral densities

\[ \tilde{S}_{vv}(jf) = \tilde{S}_{xx}(jf) + \tilde{S}_{yy}(jf) + 2 \cdot \text{Re}\{\tilde{S}_{xy}(jf)\} \]
Example: Cross Spectral Densities

The Power $P = \frac{V^2(t)}{R}$ has expected value
$E[V(t)\bar{V}(t)/R] = R_{VV}(0)/R$

And in the bandwidth between $f_{low}$ and $f_{high}$,

$P = \int_{f_{low}}^{f_{high}} S_{VV}(jf)df$ ...

Integrating with respect to frequency (over whatever bandwidth is relevant) gives the total (expected) power dissipated in R.

Note that the cross-spectral density is relevant.
Our Notation for Spectral Densities and Correlations

<table>
<thead>
<tr>
<th>Random Process</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>function of time</strong></td>
<td>$V(t)$</td>
</tr>
<tr>
<td><strong>function of frequency</strong></td>
<td>$V(jf), V(j\omega)$</td>
</tr>
<tr>
<td><strong>autocorrelation function</strong></td>
<td>$R_{VV}(\tau) = E[V(t)V(t+\tau)]$</td>
</tr>
<tr>
<td></td>
<td>$S_{VV}(j\omega) = \mathcal{F}[R_{VV}(\tau)]$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{S}<em>{VV}(jf) = 2S</em>{VV}(j\omega/2\pi)$</td>
</tr>
<tr>
<td><strong>power spectral density</strong></td>
<td>$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$</td>
</tr>
<tr>
<td></td>
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<tr>
<td></td>
<td>$\tilde{S}<em>{XY}(jf) = 2S</em>{XY}(j\omega/2\pi)$</td>
</tr>
</tbody>
</table>

When context makes it clear whether $\nu = \nu(t)$ or $\nu = \nu(j\omega)$, we can simply write $\nu$.

For stationary ergodic processes

$S_{VV}(j\omega) = S_{\nu\nu}(j\omega) = \nu(j\omega)\nu^*(j\omega)$ and $S_{XY}(j\omega) = S_{xy}(j\omega) = x(j\omega)y^*(j\omega)$
Example: Noise passing through filters & linear electrical networks

If the filter has impulse response \( h(t) \) and transfer function \( h(j\omega) \), then for any \( v_{\text{in}}(t) \rightarrow v_{\text{out}}(t) \), \( v_{\text{out}}(j\omega) = h(j\omega)v_{\text{in}}(j\omega) \)

So

\[
v_{\text{out}}(j\omega)v_{\text{out}}^*(j\omega) = h(j\omega)v_{\text{in}}(j\omega)h^*(j\omega)v_{\text{in}}^*(j\omega)
\]

\[
S_{v_{\text{out}}v_{\text{out}}}(j\omega) = \|h(j\omega)\|^2 S_{v_{\text{in}}v_{\text{in}}}(j\omega)
\]

\[
S_{V_{\text{out}}V_{\text{out}}}(j\omega) = \|h(j\omega)\|^2 S_{V_{\text{in}}V_{\text{in}}}(j\omega)
\]

\[
v_{\text{out}}(j\omega)v_{\text{in}}^*(j\omega) = h(j\omega)v_{\text{in}}(j\omega)v_{\text{in}}^*(j\omega)
\]

\[
S_{v_{\text{out}}v_{\text{in}}}(j\omega) = h(j\omega)S_{v_{\text{in}}v_{\text{in}}}(j\omega)
\]

\[
S_{V_{\text{out}}V_{\text{in}}}(j\omega) = h(j\omega)S_{V_{\text{in}}V_{\text{in}}}(j\omega)
\]

It is trivial to change to single-sided Hz-based spectral densities.