Transactions Briefs

Extension of the Cochrun–Grabel Method to Allow for Mutual Inductances

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Abstract—The Cochrun–Grabel (C–G) method, an algorithm for finding the characteristic polynomial of a circuit containing reactances, has so far been restricted to circuits not employing mutual inductances.

In this paper we present an intuitive, yet rigorous, proof of the Cochrun–Grabel method for a general RLC circuit, and we extend the method to allow the analysis of an RLC circuit containing mutual inductances.

Index Terms—Circuit theory, Cochrun–Grabel method, method of time constants, mutual inductances.

I. INTRODUCTION

The Cochrun–Grabel (C–G) approach [1] to the problem of finding the characteristic polynomial of a reactive circuit has two main features: first, it only requires the analysis of frequency-independent subcircuits derived from the circuit under study and second, it shows clearly how the polynomial coefficients depend on the circuit reactances.

The method may also be used for estimating the dominant pole of a multistage amplifier (see, e.g., [2]–[4]) where typically only the coefficient of the linear s term in the polynomial is calculated. The approximation so obtained is often very close to the actual pole value.

Extensions to the method have been presented in [5]–[7]. In this paper we give a more physical, yet rigorous, proof of the method, 1 and we extend it to handle general RLC circuits with mutual inductances.

II. THE COCHRUN–GRABEL METHOD

It is well known that the characteristic polynomial of an RC circuit is a linear function of each capacitive admittance. The characteristic polynomial of an RLC circuit is also a linear function, but of each inductive impedance. To see this it is sufficient to apply the modified nodal analysis [8] to a circuit, resulting in the equation

\[
\begin{bmatrix}
    G + Cs & \cdots & 0 \\
    \cdots & \cdots & \cdots \\
    0 & \cdots & R + Ls
\end{bmatrix}
\begin{bmatrix}
    V_0 \\
    \cdots \\
    V_n
\end{bmatrix}
= \begin{bmatrix}
    \mathbf{i}_0 \\
    \cdots \\
    \mathbf{i}_n
\end{bmatrix}
\]

(1)

where \(s\) is the complex frequency as usual. It is clear that the role of the \(Cs\)’s and the \(Ls\)’s in the determinant expansion for the matrix in (1) is identical.

The characteristic polynomial [i.e., the determinant of the matrix in (1)] can be written in general as

\[
p(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n.
\]

(2)

In the following, we will consider the normalized polynomial:

\[
p(s) = \frac{p(s)}{a_0} = 1 + a_1 s + a_2 s^2 + \cdots + a_n s^n.
\]

(3)

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1 The original proof was limited to RC and RL circuits only.

We have seen that \(p(s)\) is a linear function of the circuit reactances. Therefore, we can write its coefficients as

\[
a_i = \sum a_{i, \mathbf{X}_1, \ldots, \mathbf{X}_m} X_1 \cdots X_m
\]

(4)

with \(X\) being either \(L\) or \(C\). The summation is performed over the \(\binom{n}{i}\) combinations of \(i\) reactive elements chosen from the \(n\) available reactive elements.

The main point upon which our construction of the C–G algorithm hinges is that \(a_{i, \mathbf{X}_1, \ldots, \mathbf{X}_m}\) is independent of the \(X\) values (by the very definition of \(a_{i, \mathbf{X}_1, \ldots, \mathbf{X}_m}\) so that \(a_{i, \mathbf{X}_1, \ldots, \mathbf{X}_m}\) can be determined by assigning arbitrary values to each \(X\). In particular, these values can be the limit values zero and infinity (for each \(a_{i, \mathbf{X}_1, \ldots, \mathbf{X}_m}\) the value assigned to each \(X\) changes in general). In turn, the circuit becomes purely resistive when each \(X\) has the value zero or infinity. We will use this fact to show that the general expression for \(a_{i, \mathbf{X}_1, \ldots, \mathbf{X}_m}\) is a product of resistances and conductances obtained from a resistive subset of the circuit under study

\[
a_{i, \mathbf{X}_1, \ldots, \mathbf{X}_m} = \frac{\beta_0 X_{\mathbf{X}_1} \cdots X_{\mathbf{X}_m}}{\beta_X X_{\mathbf{X}_1} \cdots X_{\mathbf{X}_m}}
\]

(5)

where \(\beta_X\) is a resistance (conductance) if \(X\) is a capacitance (inductance). The superscripts \(X_{\mathbf{X}_1} \cdots X_{\mathbf{X}_m}\) indicate that \(\beta_X\) is to be calculated with \(X_{\mathbf{X}_1} \cdots X_{\mathbf{X}_m}\) short circuited (open circuited) if they are capacitances (inductances), while the remaining capacitances (inductances) are open circuited (short circuited).

Equation (5) enables (4) to be expressed as a sum of products of time constants, which is the form most usually taken by the C–G theorem

\[
a_i = \sum \left( \frac{\beta_0 X_{\mathbf{X}_1}}{\beta_X X_{\mathbf{X}_1}} \right) \left( \frac{\beta_0 X_{\mathbf{X}_2}}{\beta_X X_{\mathbf{X}_2}} \right) \cdots \left( \frac{\beta_0 X_{\mathbf{X}_m}}{\beta_X X_{\mathbf{X}_m}} \right) X_{\mathbf{X}_i}.
\]

(6)

A. Proof of the Method

In the following, a network with only one capacitor and one inductor is considered, but it will be clear that the proof is easily extended to the general case. The characteristic polynomial for such an LC network is, according to (3) and (4)

\[
p(s) = 1 + a_1 s + a_2 s^2 = 1 + (a_L, L + a_C, s) + a_L L C s^2.
\]

(7)

We must now supply a procedure for finding \(a_L, a_C,\) and \(a_L C.\)

We start by letting \(C \to 0.\) In this case \(p(s) \to 1 + a_L L s\) and \(a_L\) is readily found through the well-known relation \(a_L = \frac{G_L^0}{L}\) where \(G_L^0\) being the conductance seen by \(L\) when \(C\) is open circuited. Similarly, \(a_C\) is equal to \(R_C^0\) the resistance seen by \(C\) when \(L\) is short circuited (since in this case \(L \to 0\)).

It remains to derive \(a_L C.\) We start by noticing that if we let \(L = kC,\) \(k\) in the appropriate units, then for \(k \to \infty\) the value \(1/a_L L \) is a root of \(p(s)\)

\[
\lim_{k \to \infty} \frac{1}{a_L L} = \lim_{k \to \infty} \left( \frac{1}{a_L L + a_C C + a_L C L C} \right) = \lim_{k \to \infty} \frac{k}{k} = 0.
\]

We rewrite \(p(s)\) as an explicit function of its roots

\[
p(s) = (1 + \tau_1 s)(1 + \tau_2 s) = 1 + (\tau_1 + \tau_2)s + \tau_1 \tau_2 s^2.
\]

(8)
Since \(-1/\alpha_L L\) is a root of \(p(s)\) we can set \(\tau_1 = \alpha_L L\). From (7) and (8) we obtain then
\[
\tau_2 = \frac{\tau_1 \tau_2}{\tau_1} = \frac{\alpha_L L C}{\alpha_L L} = \frac{\alpha_L C}{\alpha_L}.
\]
Thus, \(\tau_2\) depends on \(C\) but not on \(L\). Furthermore, since the condition \(k \to \infty\) implies \(L \to \infty\) for a finite value of \(C\), \(C\) sees a circuit where \(L\) can be replaced by an open circuit, so that we can write \(\tau_2\) as
\[
\tau_2 = R_C^L C
\]
with \(R_C^L\) defined as the resistance seen by \(C\) with \(L\) open circuited. From (9) and (10) we arrive at the final expression for \(\alpha_L C\)
\[
\alpha_L C = \frac{\alpha_L R_C^L}{\alpha_L} = G_{22} R_C^L.
\]
By interchanging the roles of \(L\) and \(C\) we obtain the alternative relation \(\alpha_C L = R_C^L G_{22}^L\).

The above procedure is straightforwardly generalized to a network with \(n\) reactive elements, for which (5) holds.

### III. MUTUAL INDUCTANCES

The C–G method cannot yet handle mutual inductances, and we want to extend the algorithm to allow for the presence of such components in the network (Fig. 1). Let the mutual inductance \(M\) be modeled as two current-controlled voltage sources \(M_{ij2}\) and \(M_{ij1}\). The characteristic polynomial is then linear in \(M_1\) and \(M_2\) as (11) clearly shows
\[
\begin{pmatrix}
L_1 s & M_1 s & -1 & 1 & 0 & 0 \\
M_2 s & L_2 s & 0 & 0 & -1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\begin{pmatrix}
i_1 \\
i_2 \\
v_{1u} \\
v_{1d} \\
v_{2u} \\
v_{2d}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\]
(11)

Letting every \(X\) except \(M_1\) approach zero, we obtain \(p(s) = 1 + \alpha_M M_1 s\). On the other hand, (2) yields
\[
p'(s) = \Delta_{12} M_1 s + \Delta
\]
(12)
where \(\Delta\) is the determinant of the matrix in (11) and \(\Delta_{12}\) is the determinant of the matrix obtained by removing the first row and the second column from the matrix in (11). Both determinants are calculated with every \(X\) set to zero. Equation (12) yields \(\alpha_M M_1 = \Delta_{12}/\Delta\). Since \(\Delta/\Delta_{12} = V_{1u}/R_{12}\) for the circuit in Fig. 1(c), we obtain the relation \(\alpha_M M_1 = 1/R_{12}^0\), \(R_{12}^0\) being the transresistance \(V_{1u}/i_2\). The transresistances \(R_{12}^{M_{ij1}-M_{ij2}}\) are determined in the same way; however, if only one or more mutual inductance values approach infinity (e.g., one or more \(X_{ij} = M_{ij}\)) we cannot proceed so easily as for \(X_{ij} = L_{ij}, C_{ij}\), where the branch containing \(L_{ij}\) (\(C_{ij}\)) simply becomes an open circuit (short circuit). Referring to Fig. 2(a)–(b), if \(M_1 \to \infty\), then \(M_1 i_j \to \infty\) as well unless \(i_j \to 0\). Since no voltage in the circuit can grow indefinitely, \(i_j\) does tend to zero, and the branch connecting \(V_{ju}\) and \(V_{jd}\) can be modeled as a nullator, with constitutive equations \(i_j = 0\), \(V_{ju} - V_{jd} = 0\). The branch connecting \(V_{ju}\) and \(V_{jd}\) can, in turn, be modeled as a norator (\(i_j = unknown, V_{ju} - V_{jd} = unknown\) [9], Fig. 2(c). Having so transformed the circuit, the transresistance \(R_{12}^{M_{ij1}}\) can be found by using standard circuit analysis techniques. In general, if
As an example of the method previously described, the transresistance \( R_{\text{M}_{14}}^{\text{M}_{4}} \) for the network in Fig. 3(a) will be calculated. We begin by transforming the circuit as shown in Fig. 3(b) where inductances \( L_1 \) and \( L_2 \) have been replaced by a nullator and a norator, respectively. The presence of the nullator (\( i = 0 \)) forces the current through \( R_a \) to assume value \( V_i / (PR_a) \). Since no current can flow in the nullator, the voltage at the norator output has value \( V_l R_a / (PR_a) \) from which the relation \( i_3 = V_i R_a / (PR_a) \) and \( R_{\text{M}_{14}}^{\text{M}_{4}} = V_l / (PR_l) = R_l R_d / (PR_l) \) immediately follow. Although the calculations above are rather simple, it is clear that the presence of multiple mutual inductances in the circuit can make the application of the C–G method cumbersome.

With the exception of \( R_{\text{M}_{14}}^{\text{M}_{4}} \) and \( R_{\text{M}_{34}}^{\text{M}_{4}} \), all other transresistances are directly found by inspection. Some values are listed below.

\[
\begin{align*}
R_{\text{M}_{14}}^{\text{M}_{4}} &= R_{\text{M}_{34}}^{\text{M}_{4}} = -R_a \\
R_{\text{M}_{24}}^{\text{M}_{4}} &= R_{\text{M}_{34}}^{\text{M}_{4}} = -R_a \\
R_{\text{M}_{34}}^{\text{M}_{4}} &= R_{\text{M}_{34}}^{\text{M}_{4}} = -R_a \\
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R_{\text{M}_{34}}^{\text{M}_{4}} &= R_{\text{M}_{34}}^{\text{M}_{4}} = -R_a \
\end{align*}
\]

V. Conclusion

The work has shown how the Cochrun-Grabel method may be extended to RLC circuits containing mutual inductances. The time constants associated with the mutual inductances depend on various transresistances seen by the mutual inductances. Transresistances are no harder to find than resistances unless more mutual inductances are present. In this case, calculations may become less straightforward because of the appearance of nullator–norator pairs in the circuit.

References