Handout on Fourier Transforms

This handout supplements the coverage of Fourier transforms in the text and lectures.

1 Hertz or Radians/second?

We wish to be fluent in working with both angular frequency ω in radians/second, and frequency f in Hertz, where $\omega = 2\pi f$. Working with ω is often shorter than carrying around $2\pi f$ but working with f makes the duality between the time and frequency domains more transparent. The main thing to watch out for is that $d\omega = 2\pi df$, which impacts us when doing frequency domain integrals, such as the inverse Fourier transform, frequency domain convolution (corresponding to multiplication in time), and computing Parseval's identity.

2 A physical interpretation of the Fourier transform

We begin with an important physical interpretation of the Fourier transform of the impulse response of an LTI system..

Complex Exponential through LTI system: If a complex exponential $x(t) = e^{j\omega t}$ is passed through an LTI system with impulse response $h(t)$, then the output $y(t)$ is given by

$$\begin{aligned} y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}h(\tau)d\tau \\ = e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = H(j\omega)e^{j\omega t} \end{aligned} \quad (1)$$

Transfer function: For an LTI system with impulse response $h(t)$, the Fourier transform $H(j\omega) = H(j2\pi f)$ is called the transfer function. As you have seen in circuits classes, we often plot the magnitude $|H(j\omega)|$ and phase $\arg(H(j\omega))$ as a function of frequency.

LTI systems with real-valued impulse response: For an LTI system with real-valued impulse response $h(t)$, the transfer function is conjugate symmetric: $H(-j\omega) = H^*(j\omega)$. In other words, if $H(j\omega) = Ge^{j\theta}$ (where $G \geq 0$), then $H(-j\omega) = Ge^{-j\theta}$. Now, suppose that the input to the system is a real-valued sinusoid of the form $x(t) = \cos(\omega t + \phi)$.

$$x(t) = \frac{1}{2}\left(e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right) = \frac{1}{2}e^{j\phi}e^{j\omega t} + \frac{A}{2}e^{-j\phi}e^{-j\omega t}$$

Using (1) and linearity, we obtain that the system output is given by

$$\begin{aligned} y(t) = \tfrac{1}{2}e^{j\theta}H(j\omega)e^{j\omega t} + \tfrac{1}{2}e^{-j\theta}H(-j\omega)e^{-j\omega t} = \tfrac{1}{2}e^{j\phi}Ge^{j\theta}e^{j\omega t} + \tfrac{1}{2}e^{-j\phi}Ge^{-j\theta}e^{-j\omega t} \\ = \tfrac{G}{2}\left(e^{j(\omega t+\phi+\theta)} + e^{-j(\omega t+\phi+\theta)}\right) = G\cos(\omega t + \phi + \theta) \end{aligned} \quad (2)$$

To summarize, for an LTI system with real-valued impulse response, the response to a sinusoid is a sinusoid at the same frequency, scaled by the magnitude of the transfer function, and its phase is shifted by the phase of the transfer function, evaluated at the frequency of the sinusoid.

Figure 1 is a pictorial depiction of equations (1) and (2), showing the physical significance of the transfer function of an LTI system.

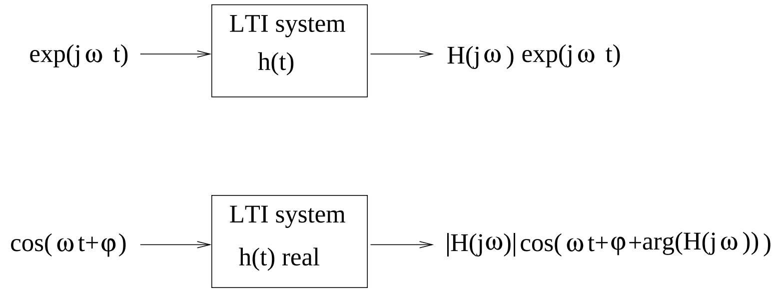

Figure 1: The physical significance of the transfer function of an LTI system.

3 What the inverse Fourier transform means

The inverse Fourier transform is given by

$$x(t) = \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} X(j2\pi f)e^{j2\pi f t}df \tag{3}$$

This can be viewed as a decomposition of $x(t)$ as a linear combination of complex exponentials $\{e^{j2\pi ft}\}$. In contrast to the Fourier series for periodic signals, where the complex exponential frequencies are integer multiples of a fundamental frequency, we need a continuum of frequencies to represent an aperiodic signal $x(t)$. Thus, instead of a sum, the linear combination is an integral, where the coefficient for a given frequency f is the Fourier transform $X(j2\pi f)$.

4 An LTI system is a filter

We can now put (1) and (3) together to gain a frequency domain understanding of what happens when a signal $x(t)$ is passed through an LTI system with impulse response $h(t)$. The signal $x(t)$ can be represented as a linear combination of complex exponentials as in (3). From (1), we know that the response to a single complex exponential $e^{j2\pi ft}$ is $H(j2\pi f)e^{j2\pi ft}$. By linearity, we obtain that the response to a linear combination of complex exponentials as in (3) is the linear combination of the responses to the individual complex exponentials. We therefore obtain that the response to $x(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} X(j2\pi f)H(j2\pi f)e^{j2\pi ft}df$$

Comparing with (3), we realize that $Y(j2\pi f) = X(j2\pi f)H(j2\pi f)$. We have therefore proven that

$$y(t) = (x * h)(t) \leftrightarrow Y(j2\pi f) = X(j2\pi f)H(j2\pi f) \tag{4}$$

Thus, we can think of each frequency propagating independently through the LTI system: the input strength at frequency f is $X(j2\pi f)$, the system transfer function, or complex gain, at that frequency is $H(j2\pi f)$, so that the output strength at frequency f is $Y(j2\pi f) = X(j2\pi f)H(j2\pi f)$. LTI systems therefore shape a signal in the frequency domain, and can therefore be termed *filters*.

Example 1: Find the output when the signal $x(t) = \text{sinc}(8t)$ is passed through a filter with impulse response $h(t) = \text{sinc}(4t)$.
Solution: The Fourier transforms occupy a finite frequency band, so it is much easier to multiply in the frequency domain than to convolve in the time domain. We have

$$\begin{aligned} x(t) = \text{sinc}(8t) \leftrightarrow X(j2\pi f) = \tfrac{1}{8}I_{[-4,4]}(f) \\ h(t) = \text{sinc}(4t) \leftrightarrow H(j2\pi f) = \tfrac{1}{4}I_{[-2,2]}(f) \end{aligned}$$

so that

$$Y(j2\pi f) = X(j2\pi f)H(j2\pi f) = \frac{1}{32} I_{[-2,2]}(f) \leftrightarrow y(t) = \frac{1}{8}\text{sinc}(4t)$$

(It is always best to draw pictures when solving such problems, even though I have not done so above.)

Example 2: Suppose that $x(t)$ is a periodic signal sent through an LTI system with impulse response $h(t)$ to obtain output $y(t)$. Then

$$x(t) = \sum_k a_k e^{jk\omega_0 t}$$

where ω_0 is the fundamental frequency. From (1) and using linearity, we obtain that the response $y(t)$ is given by

$$y(t) = \sum_k a_k H(jk\omega_0) e^{jk\omega_0 t}$$

What if both x and h are real-valued (in which case $y = x * h$ must of course be real-valued)? In this case, we know that we can write

$$x(t) = R_0 + 2\sum_{k=1}^{\infty} R_k \cos(k\omega_0 t + \phi_k)$$

where $a_k = R_k e^{j\phi_k}$. Using linearity and (2), we infer that the output

$$y(t) = R_0 H)j0) + 2\sum_{k=1}^{\infty} R_k |H(jk\omega_0)| \cos\left(k\omega_0 t + \phi_k + \arg(H(jk\omega_0)\right)$$

Example 3: In a special case of Example 2, consider a sinusoid $x(t) = 2\cos(20\pi t + \frac{\pi}{4})$ passed through an LTI system with impulse response $h(t) = \text{sinc}(40t - 1)$. We know that

$$\text{sinc}(40t) \leftrightarrow \frac{1}{40} I_{[-20,20]}(f)$$

Now, $h(t) = \text{sinc}(40(t - \frac{1}{40}))$ is a delayed version of the preceding, with a delay of 1/40, so that

$$H(j2\pi f) = \frac{1}{40} I_{[-20,20]}(f) e^{-j2\pi f(1/40)}$$

The input sinusoid is at frequency $f = 10$, for which

$$H(j2\pi f) = H(j20\pi) = \frac{1}{40} e^{-j\pi/2}$$

We can now infer from this that the output is given by

$$y(t) = 2|H(j20\pi)| \cos\left(20\pi t + \frac{\pi}{4} + \arg(H(j20\pi)\right) = \frac{1}{20}\cos(20\pi t - \frac{\pi}{4})$$

5 Time-Frequency Duality

Let $x(t) \rightarrow X(j2\pi f) = \hat{X}(f)$, where $\hat{X}$ is defined so that its argument is f rather than $2\pi f$. Thus, the Fourier and inverse Fourier transform formulas can be rewritten as

$$\begin{aligned} \hat{X}(f) &= \textstyle\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \textstyle\int_{-\infty}^{\infty} \hat{X}(f) e^{j2\pi f t} df \end{aligned} \qquad (5)$$

Expressing some common Fourier transforms in this notation, we have that

$$\begin{aligned} x(t) &= \delta(t) \leftrightarrow \hat{X} f = 1 \\ x(t) &= I_{[-\frac{T}{2},\frac{T}{2}]}(t) \leftrightarrow \hat{X} f = T\text{sinc}(fT) \end{aligned} \tag{6}$$

Now, suppose that we interchange the roles of time and frequency. Define $g(t) = \hat{X}(t)$. Then the Fourier transform $\hat{G}(f)$ is given by

$$\hat{G}(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} \hat{X}(t) e^{-j2\pi f t} dt = x(-f)$$

This is because the preceding integral is of the form of the inverse Fourier transform, except that the dummy variable of integration is now labeled t instead of f, and that the variable at which it is being evaluated is $-f$.

Statement of Duality Property: If $x(t) \leftrightarrow \hat{X}(f)$, then $g(t) = \hat{X}(t) \leftrightarrow \hat{G}(f) = x(-f)$. That is, if we switch time and frequency for a Fourier transform pair, then we should flip one of the signals around the origin.

Since even symmetric signals are unaffected by flipping, duality for such signals means that we can simply interchange time and frequency in Fourier transform pairs. For example, for the even symmetric signals in equation (6), we can infer that

$$\begin{aligned} x(t) &= 1 \leftrightarrow \hat{X}(f) = \delta(f) \\ x(t) &= B\text{sinc}(Bt) \leftrightarrow \hat{X}(f) = I_{[-\frac{B}{2},\frac{B}{2}]}(f) \end{aligned} \tag{7}$$

(we have replaced T by B simply to denote that we are now talking about bandwidth rather than the length of a time interval).

Let us now consider an example of a function that is not even. We know that

$$\delta(t - t_0) \leftrightarrow e^{-j2\pi f t_0}$$

Let us replace t by f on the left (and replace t_0 by f_0 in order to denote a frequency shift rather than a time shift). We have to replace f by $-t$ (and t_0 by f_0) on the right. This yields

$$\delta(f - f_0) \leftrightarrow e^{j2\pi f_0 t}$$

6 Physical Interpretation of Some Fourier Transform Properties

While Fourier transform properties can be derived on a purely mathematical basis, it is also important to keep in mind the physical interpretation of these properties.

Time shift: We know that a time delay of t_0 should lead to a phase delay of $2\pi f t_0$ at frequency f. This can be written as

$$x(t - t_0) \leftrightarrow X(j2\pi f) e^{-j2\pi f t_0}$$

Frequency shift: A frequency shift by f_0 should lead to a phase rotation of the form $2\pi f_0 t$. This corresponds to the property

$$X(j2\pi(f - f_0)) \leftrightarrow x(t) e^{j2\pi f_0 t}$$

Parseval's identity: As the inverse Fourier transform shows, $X(j2\pi f)$ can be viewed as a coefficient in expansion of $x(t)$ with respect to the basis formed by the complex exponentials

$\{e^{j2\pi ft}\}$. We can use our intuition for finite-dimensional vector spaces even though we actually have an uncountably infinite number of basis functions. Parseval's identity, as stated below, may therefore be viewed as saying that the inner product between two signals can be computed with respect to any basis, hence it can be computed in either the time and frequency domain.

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \int_{-\infty}^{\infty} X_1(j2\pi f)X_2^*(j2\pi f)df$$

Specializing to $x_1 = x_2 = x$, we have the result that the energy of a signal can be computed in either the time or frequency domain:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(j2\pi f)|^2 df$$

7 More examples

Some additional examples, phrased as questions and answers (sketches provided, with details to be worked out by you), are given below, to help you test your mastery of how to apply Fourier transform properties.

Question 1: What is the Fourier transform of sinc($2t$)sinc($4t$)? *Answer 1:* Multiplication in the time domain corresponds to convolution in the frequency domain. We are therefore convolving two boxes, which will result in a trapezoid. You should work out the details.

Question 2: Evaluate the integral $\int_0^\infty \text{sinc}^2(2t)\text{sinc}^2(4t)dt$.
Answer 2: By symmetry, the desired integral equals $\frac{1}{2}\int_{-\infty}^{\infty} \text{sinc}^2(2t)\text{sinc}^2(4t)dt$. Now we can use Parseval's identity and the answer to the first question to evaluate the energy in the frequency domain.

Question 3: What is the Fourier transform of $x(t) = m(t)\cos(2\pi f_0 t)$.
Answer 3: Write the cosine as a sum of complex exponentials, and note that multiplication by complex exponential corresponds to a frequency shift. You will obtain

$$X(j2\pi f) = \frac{1}{2}\left(M(j2\pi(f - f_0)) + M(j2\pi(f + f_0)\right)$$

(A similar result holds for multiplication by a sine; make sure you derive it.) If $m(t)$ is a baseband signal, and f_0 is larger than the bandwidth of m, then $x(t)$ is a bandpass signal. This property is essential to both digital and analog modulation for wireless communication, where the information to be transmitted resides in baseband signals, but the actual signal that can be transmitted must be bandpass (e.g., lying within a 20 MHz band around 2.4 GHz, as in WiFi systems).

Question 4: Draw $|X(j2\pi f)|$ and $\arg(X(j2\pi f)$ for the following waveforms $x(t)$: (a) $\text{sinc}(10t)\cos(40\pi t)$ (b) $\text{sinc}^2(10t)\sin(40\pi t)$.
Answer 4: Use Answer 3.

Question 5: Pass $x(t) = \sum_{k=-\infty}^{\infty} p(t - kT)$, where $p(t) = I_{[0,1]}$ and $T = 4$, through a filter with impulse response $h(t) = \text{sinc}^2 t$. Find an explicit expression for the output.
Answer 5: Use Example 3 in Section 4.