ECE130B Notes: Difference Equations

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The key formulas are highlighted in magenta.

1 First-order equations

In this section I describe how to find all solutions $y[n]$ of the constant coefficient linear first-order difference equation

$$y[n] + a_1\, y[n-1] = x[n],$$

with $a_1 \neq 0$.

The process is split into four major parts.

Step 1. First find the general solution of the homogeneous equation

$$y_h[n] + a_1 y_h[n-1] = 0, \qquad (1)$$

where $y_h[n]$ is also called the *zero-input response*. To do this we look for a solution of the form $y_h[n] = z^n$. On plugging this into equation (1) we see that

$$z = -a_1.$$

Therefore the general homogeneous solution is

$$y_h[n] = c\,(-a_1)^n.$$

Step 2. In this step we construct all possible impulse responses. The impulse response $h[n]$ is defined to be any solution of the equation

$$h[n] + a_1\, h[n-1] = \delta[n]. \qquad (2)$$

We observe that for $n > 0$, the impulse response satisfies the homogeneous equation

$$h[n] + a_1\, h[n-1] = 0,$$

for which we know the general solution already. Hence we can assume that

$$h[n] = c_1\,(-a_1)^n, \qquad n \geq 0,$$

for some constant c_1. Similarly when $n < 0$, the impulse response satisfies the homogenous equation, and we can again pick

$$h[n] = c_2\,(-a_1)^n, \qquad n < 0,$$

for some other constant c_2. The only value of n for which equation (2) is not satisfied at this point is $n = 0$. For this value of n the equation reads

$$\begin{aligned} h[0] + a_1\, h[-1] &= 1 \\ c_1 - c_2 &= 1. \end{aligned}$$

This last equation is called the *jump condition*. In summary, the impulse response can be written in the form

$$h[n] = \left(c_2 u[-n-1] + (1+c_2) u[n]\right)(-a_1)^n,$$

where c_2 can be chosen freely. Picking $c_2 = 0$ will give a right-going impulse response, while picking $c_2 = -1$ will give a left-going impulse response.

Step 3. The next step is to pick a *particular solution* $y_p[n]$ for the difference equation

$$y_p[n] + a_1\, y_p[n-1] = x[n].$$

The most expeditious way is to choose a suitable impulse response $h[n]$ and then pick

$$y_p[n] = x[n] * h[n].$$

Sometimes the convolution will diverge for finite values of n no matter how you pick $h[n]$. In that case you need to split $x[n] = x_1[n] + x_2[n]$, and choose two impulse responses $h_1[n]$ and $h_2[n]$, such that $y_p[n] = x_1[n] * h_1[n] + x_2[n] * h_2[n]$ can be computed with no problems.

Step 4. The general solution can be written as

$$y[n] = y_p[n] + y_h[n],$$

with one free constant in $y_h[n]$. Usually this constant is chosen to satisfy some *auxiliary conditions* that are imposed on $y[n]$ (for example, $y[0] = 0$).

2 Second-order equations

In this section I describe how to find all solutions $y[n]$ of the constant coefficient linear second-order difference equation

$$y[n] + a_1\, y[n-1] + a_2\, y[n-2] = x[n],$$

with $a_2 \neq 0$ and $x[n]$ given.

The process is split into four major parts.

Step 1. First find the two distinct[1] solutions of the homogeneous equation

$$y_h[n] + a_1 y_h[n-1] + a_2 y_h[n-2] = 0, \tag{3}$$

where $y_h[n]$ is also called the *zero-input response*. To do this we look for a solution of the form $y_h[n] = z^n$. On plugging this into equation (3) we see that z must satisfy the quadratic equation

$$z^2 + a_1 z + a_2 = 0.$$

This equation has roots (zeros) that are given by the formula

$$z_\pm = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}.$$

Exercise. Verify this.

If the quantity under the square-root (called the *discriminant*) is non-zero, then $z_+ \neq z_-$ and we can find two distinct solutions for the homogeneous equation,

$$z_-^n \qquad \text{and} \qquad z_+^n.$$

Therefore, the general solution to the homogeneous equation (3) can be taken to be

$$y_h[n] = c_- z_-^n + c_+ z_+^n,$$

[1] Officially called *linearly independent* solutions.

where c_- and c_+ are two arbitrary constants.

On the other hand, if the discriminant turns out to be zero, then z_- will be equal to z_+. In this case it turns out that we can pick the two distinct solutions of the homogeneous equation to be

$$z_-^n \qquad \text{and} \qquad nz_-^n.$$

Exercise. Check that if $z_- = z_+$ then $y_h[n] = nz_-^n$ satisfies the homogeneous equation (3).

In summary, the general homogeneous solution can be written as

$$y_h[n] = \begin{cases} c_- z_-^n + c_+ z_+^n, & \text{if } z_- \neq z_+, \\ (c_- + c_+ n) z_-^n, & \text{if } z_- = z_+, \end{cases}$$

where c_- and c_+ are two arbitrary constants.

Step 2. In this step we construct all possible impulse responses for this set of equations. The impulse response $h[n]$ is defined to be any solution of the equation

$$h[n] + a_1 h[n-1] + a_2 h[n-2] = \delta[n]. \tag{4}$$

The key idea is that any impulse response can be constructed from two pieces of zero-input responses. First, observe that if $n < 0$ then equation (4) simplifies to

$$h[n] + a_1 h[n-1] + a_2 h[n-2] = 0, \qquad n < 0,$$

which looks exactly like the homogeneous equation (3) since $\delta[n] = 0$ whenever $n < 0$. Therefore we know that when $n < 0$ any $h[n]$ must be of the form

$$h[n] = \begin{cases} c_{L-} z_-^n + c_{L+} z_+^n, & \text{if } z_- \neq z_+, \\ (c_{L-} + c_{L+} n) z_-^n, & \text{if } z_- = z_+, \end{cases} \qquad n < 0,$$

where c_{L-} and c_{L+} are two arbitrary constants.

Similarly we observe that if $n > 0$ then $h[n]$ satisfies the homogeneous equation once again, since $\delta[n] = 0$ whenever $n > 0$. But, note that the equation (4) for $n = 1$ involves both $h[0]$ and $h[-1]$. Hence, this *requires* that when $n \geq -1$ any $h[n]$ must be of the form

$$h[n] = \begin{cases} c_{R-} z_-^n + c_{R+} z_+^n, & \text{if } z_- \neq z_+, \\ (c_{R-} + c_{R+} n) z_-^n, & \text{if } z_- = z_+, \end{cases} \qquad n \geq -1,$$

where c_{R-} and c_{R+} are two other arbitrary constants.

But if we choose c_{R-} and c_{R+} independently from c_{L-} and c_{L+}, then $h[-1]$ cannot have a unique value. So we must pick these two constants such that $h[-1]$ is always given the same value. If $z_- \neq z_+$ this implies that

$$c_{L-} z_-^{-1} + c_{L+} z_+^{-1} = c_{R-} z_-^{-1} + c_{R+} z_+^{-1},$$

which places one constraint on the four constants c_{L-}, c_{L+}, c_{R-} and c_{R+}.

On the other hand if $z_- = z_+$ then the constraint on the constants becomes

$$(c_{L-} - c_{L+}) z_-^{-1} = (c_{R-} - c_{R+}) z_-^{-1},$$

which can be further simplified to

$$c_{L-} - c_{L+} = c_{R-} - c_{R+},$$

since $z_\pm \neq 0$ for a second-order difference equation.

At this point the only value of n for which equation (4) is not satisfied is $n = 0$. At this value of n the equation reads, if $z_- \neq z_+$,

$$\begin{aligned} h[0] + a_1 h[-1] + a_2 h[-2] &= 1 \\ (c_{R-} + c_{R+}) + a_1(c_{L-}z_-^{-1} + c_{L+}z_+^{-1}) + a_2(c_{L-}z_-^{-2} + c_{L+}z_+^{-2}) &= 1, \end{aligned}$$

which yields one more constraint on the four constants.

On the other hand if $z_- = z_+$ then the equation for $n = 0$ reads,

$$\begin{aligned} h[0] + a_1 h[-1] + a_2 h[-2] &= 1 \\ c_{R-} + a_1(c_{L-} - c_{L+})z_-^{-1} + a_2(c_{L-} - 2c_{L+})z_-^{-2}) &= 1. \end{aligned}$$

Let us now summarize our solutions for $h[n]$. First $h[n]$ must be of the form

$$h[n] = \begin{cases} (c_{L-}z_-^n + c_{L+}z_+^n)u[-n-1] + (c_{R-}z_-^n + c_{R+}z_+^n)u[n], & \text{if } z_- \neq z_+, \\ (c_{L-} + c_{L+}n)z_-^n u[-n-1] + (c_{R-} + c_{R+}n)z_-^n u[n], & \text{if } z_- = z_+, \end{cases} \tag{5}$$

with the understanding that there are *two* further equations that the constants $c_{L-}, c_{L+}, c_{R-}, c_{R+}$ must satisfy. These equations are called the *jump conditions* and are given by

$$\left.\begin{aligned} c_{L-}z_-^{-1} + c_{L+}z_+^{-1} - c_{R-}z_-^{-1} - c_{R+}z_+^{-1} &= 0 \\ c_{L-}(a_1 z_-^{-1} + a_2 z_-^{-2}) + c_{L+}(a_1 z_+^{-1} + a_2 z_+^{-2}) + c_{R-} + c_{R+} &= 1 \end{aligned}\right\}, \qquad \text{if } z_- \neq z_+, \tag{6}$$

and by

$$\left.\begin{aligned} c_{L-} - c_{L+} - c_{R-} + c_{R+} &= 0 \\ c_{L-}(a_1 z_-^{-1} + a_2 z_-^{-2}) - c_{L+}(a_1 z_-^{-1} + 2a_2 z_-^{-2}) + c_{R-} + c_{R+}0 &= 1 \end{aligned}\right\}, \qquad \text{if } z_- = z_+. \tag{7}$$

The two jump conditions will still leave two of the constants free in the expression for $h[n]$. Usually this freedom is used to pick either a left-going or right-going impulse response. Sometimes it must be used to pick a summable impulse response instead. For a right-going impulse response we must pick $c_{L-} = c_{L+} = 0$. Doing so, and simplifying a little bit, we obtain the following expression for the right-going impulse response

$$h_R[n] = \begin{cases} \frac{z_+^{n+1} - z_-^{n+1}}{z_+ - z_-} u[n], & \text{if } z_- \neq z_+, \\ (1+n)z_-^n u[n], & \text{if } z_- = z_+. \end{cases}$$

Exercise. Verify that $h_R[n]$ satisfies the equation

$$h_R[n] + a_1\, h_R[n-1] + a_2\, h_R[n-2] = \delta[n].$$

Step 3. The next step is to find a *particular* solution $y_p[n]$ for the difference equation

$$y_p[n] + a_1\, y_p[n-1] + a_2\, y_p[n-2] = x[n].$$

The most expeditious way is to choose a particular impulse response $h[n]$ from Step 2, and then pick

$$y_p[n] = x[n] * h[n].$$

For some problems you will find that no matter how you pick $h[n]$ the corresponding convolution will diverge for finite values of n. If that happens then you can get around it by splitting $x[n] = x_1[n] + x_2[n]$ in such a way that you can find two impulse responses $h_1[n]$ and $h_2[n]$ such that the particular solution $y_p[n] = x_1[n] * h_1[n] + x_2[n] * h_2[n]$ is well-defined for all n. However, for many problems just a left-going or right-going $h[n]$ will do the trick directly.

Step 4. The general solution can be written down as

$$y[n] = y_p[n] + y_h[n],$$

where there will be exactly two free constants (from $y_h[n]$). These two constants must be chosen to satisfy any *auxiliary* conditions that are provided.