SOLUTIONS HW-5

130-B

2) A) $y[n] - 3y[n-2] = \sin[\beta n]\, u[-n-1]$.

$y_h[n] = c_1(\sqrt{3})^n + c_2(-\sqrt{3})^n$.

$y_p[n]$:-

Take Z-Transforms on both sides:-

$$\mathcal{Z}[\sin\beta n\, u[-n-1]] = \frac{-\sin w_0\, z^{-1}}{1 - 2\cos w_0 z^{-1} + z^{-2}} \qquad |z| < 1.$$

$$\therefore\ Y(z) = \frac{-\sin w_0\, z^{-1}}{(1-3z^{-1})(1-2\cos w_0 z^{-1} + z^{-2})}$$

$$= \frac{A}{1-e^{jw_0}z^{-1}} + \frac{B}{1-e^{-jw_0}z^{-1}} + \frac{C}{1-\sqrt{3}z^{-1}} + \frac{D}{1+\sqrt{3}z^{-1}} \qquad |z|<1.$$

Finding partial fractions:-

$$A = \frac{j}{2(1-3e^{-2jw_0})} \qquad B = \frac{j}{2(1-3e^{2jw_0})} \qquad C = \frac{\frac{-\sin w_0}{\sqrt{3}}}{2\left(1-\frac{e^{jw_0}}{\sqrt{3}}\right)\left(1-\frac{e^{-jw_0}}{\sqrt{3}}\right)}$$

$$D = \frac{\frac{-\sin w_0}{\sqrt{3}}}{2\left(1+\frac{e^{jw_0}}{\sqrt{3}}\right)\left(1+\frac{e^{-jw_0}}{\sqrt{3}}\right)}.$$

$y(n) = -Ae^{j\omega_0 n} u[-n-1] - Be^{-j\omega_0 n} u[-n-1] - C(\sqrt{3})^n u[-n-1]$ ②

$- D(-\sqrt{3})^n u[-n-1]$.

as ROC is $|z| < 1$.

B) $y[n-1] + 2y[n+1] - 3y[n] = u[n+1]$.

$y_h[n]$:-

$z^{-1} + 2z - 3 = 0 \Rightarrow 2z^2 - 3z + 1 = 0$.

$2z^2 - 2z - z + 1 = 0$

$2z(z-1) - (z-1) = 0$

$z = \frac{1}{2}, 1$.

$\therefore y_h[n] = c_1\left(\frac{1}{2}\right)^n + c_2(1)^n$.

$y_p[n]$:-

$$Y(z) = \frac{\frac{z}{1-z^{-1}}}{z^{-1} + 2z - 3}$$

~~|z| > 1~~ $|z|$ unrestricted.

$$= \frac{-1}{1-z^{-1}} + \frac{1/2}{1-\frac{z^{-1}}{2}} + \frac{1}{(1-z^{-1})^2}$$

$|z|$ ~~>1~~ unstricted

(PF).

$\therefore y[n] = -u[n] + \left(\frac{1}{2}\right)^{n+1} u[n] + (n+1)u[n+1]$ for $|z| > 1$.

IIIly we get a left going solution if we take $|z| < 1$.

c) $-3y[n+1] + 2y[n+2] + y[n] = n2^{n-1}u[-n]$.

$y_h[n] = c_1\left(\frac{1}{2}\right)^n + c_2(1)^n$

$y_p[n]$:- $Y(z) = \dfrac{Z\left(n2^{n-1}u[-n]\right)}{2z^2-3z+1}$

$$Z\left(n2^{n-1}u[-n]\right) = \frac{-z^{-1}}{(1-2z^{-1})^2}.$$

$$\therefore Y(z) = \frac{A}{1-2z^{-1}} + \frac{B}{(1-2z^{-1})^2} + \frac{C}{2-z^{-1}} + \frac{D}{1-z^{-1}} \quad . \ |z|<2.$$

$A = \frac{13}{18}$, $B = -\frac{1}{6}$, $C = \frac{8}{9}$, $D = -1$.

$$\therefore y[n] = -A\,2^n u[-n-1] - B(n+1)2^n u[-n-2] - \frac{C}{2}\left[\left(\tfrac{1}{2}\right)^n u[-n-1]\right]$$
$$- D(1)^n u[-n-1]$$

D) $y[n-2] + 2y[n-1] + y[n] = 3^{n-1}$

3 methods to solve this problem:-

(i) $3^{n-1} = 3^{n-1} u[n] + 3^{n-1} u[-n-1]$.

$= x_1[n] + x_2[n]$.

$\longrightarrow y_{P_1}[n] + y_{P_2}[n]$.

Solve for $y_{P_1}[n]$ and $y_{P_2}[n]$ independently and add them to get $y_P[n]$.

It turns out:- $y_{P_1}[n] = \frac{3}{16} 3^n u[n] + \frac{1}{16} u[n] + \frac{1}{12} n u[n]$.

Observe that it is right sided.

$y_{P_2}[n] = \frac{-3}{16} 3^n u[-n-1] + \frac{1}{16} u[-n-1] + \frac{1}{12} u[-n-1] u[-n-1]$.

Observe that it is left sided.

(ii) Withat using z transforms, solve these two problems independently :-

$$y[n-2] + 2y[n-1] + y[n] = 3^{n-1} u[n]$$

→ Pick a right going $h[n]$.

choose $C_{+1}, C_{+2}, C_{-1}, C_{-2}$ Such that $h[n]$ is right going.

$$y[n-2] + 2y[n-1] + y[n] = 3^{n-1} u[-n-1]$$

→ Pick a left going $h[n]$.

finally $y_p[n] = y_{p_1}[n] + y_{p_2}[n]$.

(iii) INTERESTING METHOD :-

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{(z^{-1}+1)^2}$$

We know that if z^n is input to a system, $H(z) \times z^n$ is the output. Hence if 3^n is input to the system, $H(3) \times 3^n$ is the output. If you input $\left(\frac{1}{3}\right) 3^n$ to the system, $\frac{1}{3} H(3) \times 3^n$ is the output $= \frac{3}{16} \times 3^n$.

3) $H_1(z) = \dfrac{1}{H_2(z)} = \dfrac{Y(z)}{X(z)}$

Hence, no. of poles of $H_1(z)$ = no. of zeros of $H_1(z)$.

For stability and causality, All poles should be inside unit circle. Stability of $H_2(z)$ => all zeros of $H_1(z)$ should be inside the unit circle.

Hence, all poles and zeros of $H_1(z)$ must be inside the unit circle.

4) $y[n] + \frac{10}{3} y[n-1] + y[n-2] = x[n] + x[n-1]$.

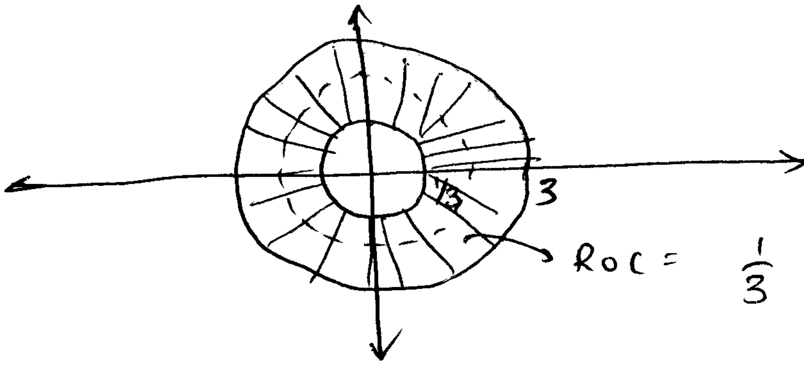
If $x[n] = u[n]$.

$$Y(z) = \frac{\dfrac{1}{1-z^{-1}} + \dfrac{z^{-1}}{1-z^{-1}}}{\left(1 + \frac{10}{3} z^{-1} + z^{-2}\right)} = \frac{1 + z^{-1}}{\left(1-z^{-1}\right)\left(1+\frac{z^{-1}}{3}\right)\left(1+3z^{-1}\right)}$$

$$= \frac{3/8}{1-z^{-1}} + \frac{1/16}{1+\frac{z^{-1}}{3}} + \frac{9/16}{1+3z^{-1}}$$

ROC for $x[n]$ = $|z| > 1$.

a) System is stable :-



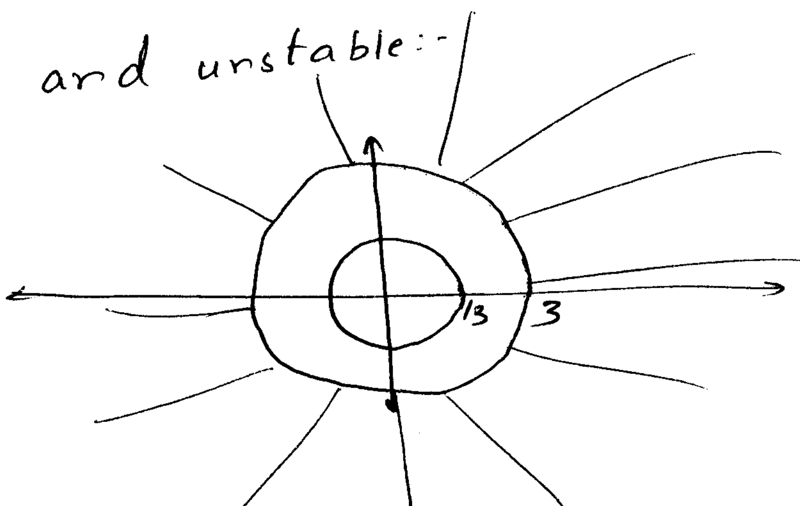
Roc = $\frac{1}{3} < |z| < 3$.

~~y[n]~~ $\therefore$ Roc = $\left(\frac{1}{3} < |z| < 3\right) \underbrace{\cap}_{\text{Intersection}} (|z| > 1)$

$= 1 < |z| < 3$.

$\therefore y[n] = \frac{3}{8} u[n] + \frac{1}{16} \left(\frac{-1}{3}\right)^n u[n] - \frac{9}{16} (-3)^n u[n]$.

b) causal and unstable :-

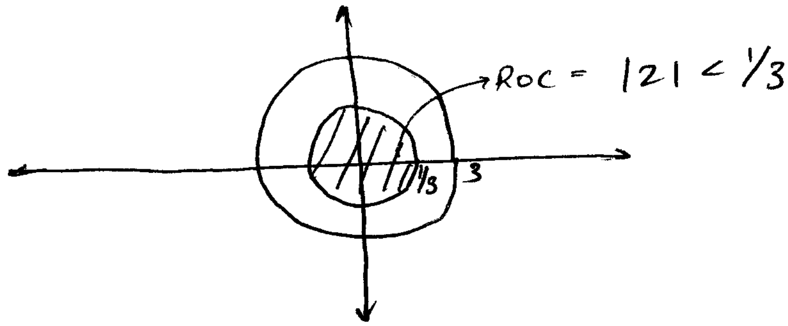


Roc = $|z| > 3$.

$\therefore$ Roc = $(|z| > 3) \cap (|z| > 1) = |z| > 3$.

$\therefore y[n] = \frac{3}{8} u[n] + \frac{1}{16} \left(\frac{-1}{3}\right)^n u[n] + \frac{9}{16} (-3)^n u[n]$.

(c) Anticausal and unstable :-



$$ROC = \left(|z| < \tfrac{1}{3}\right) \cap \left(|z| > 1\right) = \phi .$$

$\therefore y[n]$ does not converge.

5) $$H(z) = \frac{1 + \alpha z^{-1}}{1 - \alpha z^{-1}}$$

For system to be stable and causal, all poles should be inside unit circle. Hence, $|\alpha| < 1$. Also if $\alpha = -\alpha$, $H(z) = 1$, and system is stable and causal. Hence either $|\alpha| < 1$ or $\alpha = -\alpha$ //

6) $$H(z) = \frac{1 - \beta z^{-1}}{3 - 10z^{-1} + 3z^{-2}} = \frac{1 - \beta z^{-1}}{(1 - 3z^{-1})(3 - z^{-1})}$$

$\therefore$ If $\beta = 3$, poles at $z = 1/3 \Rightarrow$ system is stable and causal.

Hence $\beta = 3$.