Homework #1. #2

2. homogeneous Helmholtz's equation is

$\nabla^2 \vec{E} + k^2 \vec{E} = 0$.

$k^2 = w^2 u \varepsilon$. from (8-22),

$\nabla^2 \vec{E} = \nabla^2 \vec{E}(x,y,z) = \nabla^2 (\vec{E}_o \cdot e^{-jk_x x - jk_y y - jk_z z})$.

$= \hat{a}_x \cdot \nabla^2 E_{ox} e^{-jk_x x - jk_y y - jk_z z}$.

$+ \hat{a}_y \cdot \nabla^2 E_{oy} e^{-jk_x x - jk_y y - jk_z z}$.

$+ \hat{a}_z \nabla^2 E_{oz} e^{-jk_x x - jk_y y - j \cdot k_z z}$.

$= \hat{a}_x E_{ox} (-k_x^2 - k_y^2 - k_z^2) \cdot e^{-jk_x x - jk_y y - jk_z z}$.

$+ \hat{a}_y E_{oy} (-k_x^2 - k_y^2 - k_z^2) \cdot e^{-jk_x x - jk_y y - jk_z z}$.

$+ \hat{a}_z E_{oz} (-k_x^2 - k_y^2 - k_z^2) \cdot e^{-jk_x x - jk_y y - jk_z z}$.

$= -(k_x^2 + k_y^2 + k_z^2) \cdot e^{-jk_x x - jk_y y - jk_z z} (\hat{a}_x E_{xo} + \hat{a}_y E_{yo} + \hat{a}_z E$

$= -(k_x^2 + k_y^2 + k_z^2) \cdot \vec{E}_o \cdot e^{-jk_x x - jk_y y - jk_z z}$

$= -k^2 \cdot \vec{E}$ (from (8-23)).

so

$\nabla^2 \vec{E} + k^2 \vec{E} = -k^2 \cdot \vec{E} + k^2 \vec{E} = 0$

Notation : $\vec{E}$ means vector E

$\vec{E}_o = \hat{a}_x E_{ox} + \hat{a}_y E_{oy} + \hat{a}_z E_{oz}$.

$\hat{a}_x, \hat{a}_y, \hat{a}_z$ are unit vectors

3. Assume that the vehicle moves with a velocity u in $(+z)$ direction, which is the direction of wave propagation.

(a). for the vehicle.

if, $\vec{E}_{in} = \hat{a}_x E_o e^{j(wt - kz)}$

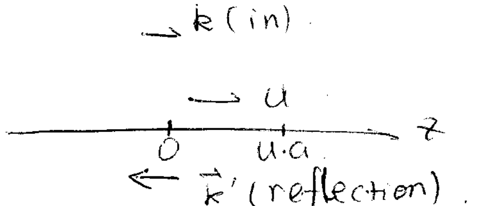
then, $\vec{E}_{ref} = -\hat{a}_x \cdot E_o \cdot e^{j(w't + k'z)}$

$\Bigg\}$ perfectly reflecting $\Rightarrow$ the amplitude (E_o) keeps unchanged.

Note, $w \neq w'$. $k \neq k'$

Consider the reflecting surface, we have.

$\vec{E}_{in} + \vec{E}_{ref} = 0$. at the reflecting surface



(1) $t = 0$

$\vec{E}_{in} = \hat{a}_x E_o \cdot e^{j(-kz)}$

$\vec{E}_{ref} = -\hat{a}_x E_o \, e^{j(+k'z)}$

$\vec{E}_{in}(z) + \vec{E}_{ref}(z) = 0$. at the reflecting surface

so, $z = 0$

(2) $t = a$ the reflecting surface is located at

$z = u \cdot t = u \cdot a$

$\vec{E}_{in}(ua) + \vec{E}_{ref}(ua) = 0$

so that, $wa - ku \cdot a = w'a + k'ua$.

$\underset{\sim\sim\sim}{w - ku = w' + k'u}$

since $k = \frac{w}{c}$, $k' = \frac{w'}{c}$.

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$$w - ku = w' + k'u.$$

$$w - w' = \frac{u}{c}(w + w').$$

$$(1 - \frac{u}{c})\, w = (1 + \frac{u}{c}) w'$$

$$\frac{w'}{w} = \frac{2\pi f'}{2\pi f} = \frac{f'}{f} = \frac{1 - \frac{u}{c}}{1 + \frac{u}{c}} = \frac{(1 - \frac{u}{c})^2}{(1 - \frac{u^2}{c^2})}$$

$$= \frac{1 - \frac{2u}{c} + \frac{u^2}{c^2}}{1 - \frac{u^2}{c^2}}.$$

because $u \ll c$. $\frac{u}{c} \ll 1$,

$$\frac{f'}{f} = \frac{1 - \frac{2u}{c}}{1} = 1 - \frac{2u}{c}.$$

$$\Delta f = f' - f = f(1 - \frac{2u}{c}) - f = f \cdot (-\frac{2u}{c}).$$

(b). $\Delta f = 2.33$ kHz

$f = 10.5$ GHz

$$u = -\frac{\Delta f \times c}{2f} = -\frac{2.33 \times 10^3 \times 3 \times 10^8}{2 \times 10.5 \times 10^9}$$

$= -33.28$ m/s $= -119.8$ km/hr $= 74.6$ (miles/hr).

4.

$$\vec{E} = \vec{E}_0 e^{-j\vec{k}\cdot\vec{R}}$$

$$\vec{H} = \vec{H}_0 \cdot e^{-j\vec{k}\cdot\vec{R}}$$

$\vec{E}_0$, $\vec{H}_0$ are constant vectors.

consider $\nabla\times\vec{E}$.

$$\nabla\times\vec{E} = \nabla\times\left(\vec{E}_0\cdot e^{-j\vec{k}\cdot\vec{R}}\right)$$

assume: $\vec{E}_0 = \hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z$

E_x, E_y, E_z are constant scalars.

$$\nabla\times\left(\vec{E}_0\cdot e^{-j\vec{k}\cdot\vec{R}}\right) = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x e^{-j\vec{k}\cdot\vec{R}} & E_y e^{-j\vec{k}\cdot\vec{R}} & E_z e^{-j\vec{k}\cdot\vec{R}} \end{vmatrix}$$

$$= \hat{a}_x\left[E_z\cdot(-jk_y)\cdot e^{-j\vec{k}\cdot\vec{R}} - E_y\cdot(-jk_z)\cdot e^{-j\vec{k}\cdot\vec{R}}\right]$$

$$+ \hat{a}_y\left[E_x(-jk_z)\,e^{-j\vec{k}\cdot\vec{R}} - E_z(-jk_x)\cdot e^{-j\vec{k}\cdot\vec{R}}\right]$$

$$+ \hat{a}_z\left[E_y(-jk_x)\,e^{-j\vec{k}\cdot\vec{R}} - E_x(-jk_y)\cdot e^{-j\vec{k}\cdot\vec{R}}\right]$$

$$= \left(-j\cdot e^{-j\vec{k}\cdot\vec{R}}\right)\cdot\left[\hat{a}_x(E_z k_y - E_y k_z) + \hat{a}_y(E_x k_z - E_z k_x) + \hat{a}_z(E_y k_x - E_x k_y)\right]$$

$$= \left(-j\cdot e^{-j\vec{k}\cdot\vec{R}}\right)\cdot\left(\vec{k}\times\vec{E}_0\right)$$

$$= -j\cdot\vec{k}\times\left(\vec{E}_0\cdot e^{-j\vec{k}\cdot\vec{R}}\right)$$

$$= -j\cdot\vec{k}\times\vec{E}$$

In the same way, $\nabla\times\vec{H} = -j\vec{k}\times\vec{H}$.

$$\nabla \cdot \vec{E} = \nabla \cdot (\vec{E}_0 e^{-j\vec{k}\cdot\vec{R}})$$

$$= E_x \cdot (-jk_x) e^{-j\vec{k}\cdot\vec{R}} + E_y(-jk_y) \cdot e^{-j\vec{k}\cdot\vec{R}} + E_z(-jk_z) e^{-j\vec{k}\cdot}$$

$$= -j \cdot \vec{k} \cdot \vec{E}_0 e^{-j\vec{k}\cdot\vec{R}}$$

$$= -j(\vec{k}\cdot\vec{E}).$$

Also,

$$\nabla \cdot \vec{H} = -j\vec{k}\cdot\vec{H}$$

Now consider Maxwell's equations.

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} = -jw\vec{B} = -jw \cdot u\vec{H}$$

$$\nabla \times \vec{E} = -j\vec{k}\times\vec{E}$$

so that $\quad -j\vec{k}\times\vec{E} = -jwu\vec{H}$

$$\vec{k}\times\vec{E} = wu\vec{H}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D} = \vec{J} + (+jw)\varepsilon\vec{E}.$$

$$= 0 + jw\varepsilon\vec{E} \quad (\text{source-free})$$

$$\nabla\times\vec{H} = -j\vec{k}\times\vec{H}$$

so that $\quad -j\vec{k}\times\vec{H} = +jw\varepsilon\vec{E}.$

$$\vec{k}\times\vec{H} = -w\varepsilon\vec{E}$$

$$\nabla\cdot\vec{E} = -j\vec{k}\cdot\vec{E} = 0. \quad \Rightarrow \quad \vec{k}\cdot\vec{E} = 0.$$

$$\nabla\cdot\vec{H} = -j\vec{k}\cdot\vec{H} = 0 \quad \Rightarrow \quad \vec{k}\cdot\vec{H} = 0.$$

8-7. Consider $E(0,t) = \hat{a}_x E_{10} \cdot \sin(wt) + \hat{a}_y E_{20} \cdot \sin(wt + \psi)$

$$= \hat{a}_x E_x + \hat{a}_y E_y$$

$$E_x = E_{10} \sin(wt)$$

$$E_y = E_{20} \sin(wt + \psi) = E_{20} \cdot [\sin(wt) \cdot \cos\psi + \sin\psi \cos(w$$

then,

$$\frac{E_x}{E_{10}} = \sin(wt) \qquad (1)$$

$$\frac{E_y}{E_{20}} = \sin(wt) \cdot \cos\psi + \sin\psi \cdot \cos(wt) \qquad (2)$$

$$= \frac{E_x}{E_{10}} \cdot \cos\psi + \sin\psi \cdot \cos(wt)$$

so, from (2)

$$\frac{E_y}{E_{20}} - \frac{E_x}{E_{10}} \cdot \cos\psi = \sin\psi \cdot \cos(wt)$$

$$\left(\frac{E_y}{E_{20}} - \frac{E_x}{E_{10}} \cdot \cos\psi\right)^2 = \sin^2\psi \cdot \cos^2(wt)$$

$$= \sin^2\psi \cdot (1 - \sin^2(wt))$$

$$= \sin^2\psi \cdot \left(1 - \left(\frac{E_x}{E_{10}}\right)^2\right)$$

expand the left-side,

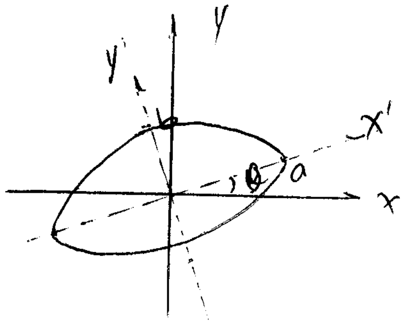
$$\left(\frac{E_y}{E_{20}}\right)^2 + \left(\frac{E_x}{E_{10}}\right)^2 \cdot \cos^2\psi - 2\frac{E_x E_y}{E_{10} E_{20}} \cos\psi = \sin^2\psi - \left(\frac{E_x}{E_{10}}\right)^2 \sin^2\psi$$

we have,

$$\left(\frac{E_y}{E_{20}}\right)^2 + \left(\frac{E_x}{E_{10}}\right)^2 - 2\frac{E_x E_y}{E_{10} E_{20}} \cos\psi = \sin^2\psi$$

$$\left(\frac{E_x}{E_{10}\sin\psi}\right)^2 + \left(\frac{E_y}{E_{20}\sin\psi}\right)^2 - 2 \cdot \frac{E_x E_y}{E_{10} E_{20}} \frac{\cos\psi}{\sin^2\psi} = 1 \qquad (*)$$

This is already an ellipse.



To find the parameters of the polarizat ellipse, we need to rotate the coordinate axes $x-y$ counterclockwise by an angle θ, to $x'-y'$.

Assume the ellipse in terms the new coordinates is

$$\left(\frac{E_{x'}}{a}\right)^2 + \left(\frac{E_{y'}}{b}\right)^2 = 1 \qquad (3).$$

$$\begin{cases} E_{x'} = E_x \cos\theta + E_y \sin\theta \\ E_{y'} = -E_x \sin\theta + E_y \cos\theta \end{cases} \qquad (4).$$

put (4) in (3)

$$E_x^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) + E_y^2\cdot\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)$$

$$+ 2\,\frac{E_x E_y \sin\theta\cdot\cos\theta}{a^2} - 2\cdot\frac{E_x E_y \sin\theta\cdot\cos\theta}{b^2} = 1$$

so that compare with the former equation (*).

$$\begin{cases} \dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2} = \dfrac{1}{E_{10}^2\cdot\sin^2\phi} \\ \dfrac{\sin^2\theta}{a^2} + \dfrac{\cos^2\theta}{b^2} = \dfrac{1}{E_{20}^2\cdot\sin^2\phi}. \\ \sin\theta\cos\theta\cdot\left(\dfrac{1}{a^2} - \dfrac{1}{b^2}\right) = \dfrac{1}{E_{10}\cdot E_{20}}\cdot\dfrac{\cos\phi}{\sin^2\phi}. \end{cases}$$

The solutions are:

$$\theta = \frac{1}{2}\cdot\tan^{-1}\left(\frac{2E_{10}E_{20}\cos\phi}{E_{10}^2 - E_{20}^2}\right)$$

$$a = \left[\frac{2}{\frac{1}{E_{10}^2}(1+\sec(2\theta)) + \frac{1}{E_{20}^2}(1-\sec(2\theta))}\right]^{\frac{1}{2}}\cdot\sin\phi.$$

$$b = \left[\frac{2}{\frac{1}{E_{10}^2}\cdot(1-\sec(2\theta)) + \frac{1}{E_{20}^2}(1+\sec(2\theta))}\right]^{\frac{1}{2}}\sin\phi.$$

8-8. (a). Consider the phasor at $z=0$.

An elliptically polarized wave can be written

$$\vec{E} = \hat{a}_x E_1 + \hat{a}_y E_2 \cdot e^{j\alpha}.$$

Assume the Right-hand circularly polarized wave

$$\vec{E}_{rc} = E_{rc}\,(\hat{a}_x - \hat{a}_y \cdot j).$$

the Left-hand circularly polarized wave:

$$\vec{E}_{lc} = E_{lc}\,(\hat{a}_x + \hat{a}_y \cdot j).$$

if $E_{rc} = \frac{1}{2}(E_1 + jE_2 \cdot e^{j\alpha})$

$E_{lc} = \frac{1}{2}(E_1 - jE_2 \cdot e^{j\alpha})$.

then $\vec{E} = \vec{E}_{rc} + \vec{E}_{lc}$.

(b).

$$\vec{E}_{rc} = E\,(\hat{a}_x - \hat{a}_y j).$$

$$= E\,(\hat{a}_x \tfrac{1}{2} - \hat{a}_y \cdot j \cdot 2) + E\,(\hat{a}_x \tfrac{1}{2} + \hat{a}_y j)$$

These two terms are oppositely directed elliptically polarized waves.

The solution is not unique.

8-9. for a conducting media,

$$\frac{\sigma}{w} \gg \varepsilon$$

consider a general expression,

$$\gamma = \alpha + j\beta = j\,k_c = j\,w\sqrt{\mu\,\varepsilon_c}.$$

(9) from Maxwell's equations,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D} = \sigma \cdot \vec{E} + j\omega\varepsilon\vec{E}$$

$$= j\omega \cdot \left(\varepsilon - j\cdot\frac{\sigma}{\omega}\right)\vec{E} = j\cdot\omega\cdot\varepsilon_c\cdot\vec{E}.$$

$$\varepsilon_c = \varepsilon - j\cdot\frac{\sigma}{\omega}.$$

$$(\alpha + j\cdot\beta)^2 = -\omega^2\cdot\mu\cdot\varepsilon_c.$$

here, $\alpha, \beta, \omega, \mu, \varepsilon, \sigma$ are all real number,

$$\alpha^2 - \beta^2 + j\cdot 2\alpha\beta = -\omega^2\mu\cdot\left(\varepsilon - j\cdot\frac{\sigma}{\omega}\right).$$

$$= -\omega^2\mu\varepsilon + j\cdot\omega\cdot\mu\cdot\sigma$$

so,

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon. \quad ①$$

$$2\alpha\beta = \omega\mu\cdot\sigma.$$

~~then~~ and

$$(\alpha^2 + \beta^2) = |\alpha + j\beta|^2 = |-\omega^2\mu\varepsilon_c|.$$

$$= |\omega^2\mu\varepsilon - j\omega\mu\sigma|$$

$$= \sqrt{\omega^4\mu^2\varepsilon^2 + \omega^2\mu^2\sigma^2} = \omega^2\mu\varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} \quad ②$$

from ①, ②.

$$\alpha^2 = \frac{\omega^2\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right).$$

$$\beta^2 = \frac{\omega^2\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right).$$

we can show that

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}\cdot\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)^{\frac{1}{2}}.$$

$$\beta = \omega\cdot\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)^{\frac{1}{2}}.$$