Homework 8

1. From Maxwell's Equations,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad (1)$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \quad (2)$$

$$\nabla = \nabla_T + \nabla_z = \nabla_T + \hat{a}_z \frac{\partial}{\partial z}$$

$$\vec{E} = \vec{E}_T + \hat{a}_z E_z \qquad \vec{H} = \vec{H}_T + \hat{a}_z H_z$$

Plug in (1), (2)

$$(\nabla_T + \hat{a}_z \tfrac{\partial}{\partial z}) \times (\vec{E}_T + \hat{a}_z E_z) = -j\omega\mu(\vec{H}_T + \hat{a}_z H_z)$$

$$\nabla_T \times \vec{E}_T + \gamma \vec{E}_T \times \hat{a}_z + \nabla_T \times (\hat{a}_z E_z) = -j(\omega\mu)(\vec{H}_T + \hat{a}_z H_z)$$

Since the wave has an arbitrary cross section,

$$\nabla_T \times \vec{E}_T = 0$$

So that

$$\nabla_T E_z \times \hat{a}_z + \gamma \vec{E}_T \times \hat{a}_z = j\omega\mu(\vec{H}_T + \hat{a}_z H_z)$$

Consider xy-plane,

$$\nabla_T E_z \times \hat{a}_z + \gamma \vec{E}_T \times \hat{a}_z = -j\omega\mu\vec{H}_T \quad (3)$$

Also, from (2),

$$\nabla_T H_z \times \hat{a}_z + \gamma \vec{H}_T \times \hat{a}_z = j\omega\varepsilon\vec{E}_T \quad (4)$$

From (3), (4),

$$-j\omega\mu(\nabla_T H_z \times \hat{a}_z) + \gamma(-j\omega\mu\vec{H}_T) \times \hat{a}_z = \omega^2\mu\varepsilon\vec{E}_T$$

$$-j\omega\mu(\nabla_T H_z \times \hat{a}_z) + \gamma(\nabla_T E_z \times \hat{a}_z + \gamma\vec{E}_T \times \hat{a}_z) \times \hat{a}_z = \omega^2\mu\varepsilon\vec{E}_T$$

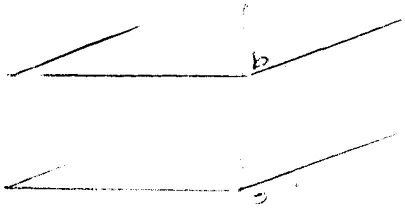
$$j\omega\mu\, \hat{a}_z \times \nabla_T H_z - \gamma\nabla_T E_z - \gamma^2\vec{E}_T = \omega^2\mu\varepsilon\vec{E}_T$$

$$(\omega^2\mu\varepsilon + \gamma^2)\vec{E}_T = h^2\vec{E}_T = j\omega\mu\,\hat{a}_z \times \nabla_T H_z - \gamma\nabla_T E_z$$

$$\vec{E}_T = -\frac{1}{h^2}(\gamma\nabla_T E_z - j\omega\mu\,\hat{a}_z \times \nabla_T H_z)$$

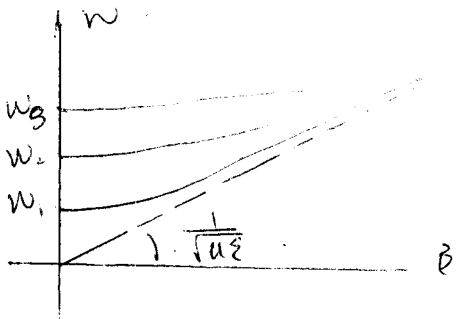
Similarly, $\vec{H}_T = -\frac{1}{h^2}(\gamma\nabla_T E_z + j\omega\varepsilon\,\hat{a}_z \times \nabla_T E_z)$

3.



$$\omega^2\mu\varepsilon - \beta^2 = \left(\frac{n\pi}{b}\right)^2$$

$$\omega_1 = \frac{\pi}{b\sqrt{\mu\varepsilon}}, \quad \omega_2 = \frac{2\pi}{b\sqrt{\mu\varepsilon}} \quad \omega_3 = \frac{3\pi}{b\sqrt{\mu\varepsilon}}.$$



(a). b affects the cutoff frequency
μ, ε affect the cutoff frequency and the slope ~~of~~ of the $\omega - \beta$ curve

As $\beta \to +\infty$, $\omega \to +\infty$. all curves turn out to be $\omega^2\mu\varepsilon = \beta^2$

$$\omega = \frac{1}{\sqrt{\mu\varepsilon}}\beta.$$

(b). From Maxwell's equations, we can also get

$$\omega^2\mu\varepsilon - \beta^2 = \left(\frac{n\pi}{b}\right)^2$$

for TE modes.

So they share the same curves.

5. For TE_n mode.

$$H_z(y) = B_n \cdot \cos(n\pi y/b)$$

$$H_y(y) = \frac{r}{h} \cdot B_n \cdot \sin(n\pi y/b)$$

$$E_x(y) = \frac{j\omega\mu}{h} B_n \sin(n\pi y/b).$$

$y=0$, $\bar{J}_{s0} = \hat{a}_y \times \bar{H}(0) = \hat{a}_x B_n$

$y=b$. $\bar{J}_{sb} = (-\hat{a}_y) \times \bar{H}(b) = \hat{a}_x (-1)^{n+1} \cdot B_n = \begin{cases} \bar{J}_{s0} & n \text{ is odd} \\ -\bar{J}_{s0} & n \text{ is even} \end{cases}$

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$$\vec{P}_{av} = \frac{1}{2} Re\left(\vec{E} \times \vec{H}^*\right) = \frac{1}{2} Re\left[\hat{a}_z E_x \cdot H_y^* - \hat{a}_y E_x \cdot H_z^*\right].$$

along z-direction

$$\vec{P}_{av} \cdot \hat{a}_z = \frac{1}{2} Re\left[E_x \cdot H_y^*\right] = \frac{\omega \mu \beta}{2h^2} \cdot B_n^2 \cdot \sin^2\left(\frac{n\pi y}{b}\right).$$

$$(P_z)_{av} = \int_0^b \vec{P}_{av} \cdot \hat{a}_z \, dy = \frac{\omega \mu \beta}{4h^2} B_n^2 \quad \text{per unit width}.$$

$$(w_e)_{av} = \frac{\varepsilon}{4} Re\left[\vec{E} \cdot \vec{E}^*\right] = \frac{\varepsilon \omega^2 \mu^2}{4h^2} B_n^2 \cdot \sin^2\left(\frac{n\pi y}{b}\right).$$

$$(W_e)_{av} = \int_0^b (w_e)_{av} dy = \frac{\varepsilon \omega^2 \mu^2}{8h^2} B_n^2 = (W_m)_{av}.$$

$$u_{en} = \frac{(P_z)_{av}}{(W_e)_{av} + (W_m)_{av}} = \frac{\omega \mu \beta}{\varepsilon \omega^2 \mu^2} = \frac{\omega \beta}{k^2} = u \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

8. (a). TEM mode:

$$\beta = \omega\sqrt{\mu\varepsilon} = 314.2 \quad (rad/m)$$

$$\alpha_d = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = 1.257 \times 10^{-8} \quad (Np/m).$$

$$\alpha_c = \frac{1}{b}\sqrt{\frac{\pi f \varepsilon}{\sigma_c}} = 2.076 \times 10^{-3} \quad (Np/m)$$

$$u_p = u_g = u = \frac{1}{\sqrt{\mu\varepsilon}} = 2 \times 10^8 \quad (m/s).$$

$$\lambda_g = \lambda = \frac{u}{f} = 2 \times 10^{-2} \quad (m).$$

(b). TM_1 mode. $f_c = \frac{1}{2b\sqrt{\mu\varepsilon}} = 2 \times 10^9$ Hz

$$\beta = \omega\sqrt{\mu\varepsilon} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 307.8 \quad (rad/m).$$

$$\alpha_d = \frac{\sigma\eta}{2\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = 1.283 \times 10^{-8} \quad (Np/m).$$

$$\alpha_c = \frac{2R_s}{\eta \cdot b \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = 4.238 \times 10^{-3} \quad (Np/m)$$

$$u_p = u \cdot \frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = 2.041 \times 10^8 \ (m/s).$$

$$u_g = u \cdot \sqrt{1-\left(\frac{f_c}{f}\right)^2} = 1.960 \times 10^8 \ (m/s).$$

$$\lambda_g = \lambda \cdot \frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = 2.041 \times 10^{-2} \ (m).$$

TM_2 mode.

(c). $$f_c = \frac{1}{b\sqrt{\mu\varepsilon}} = 4 \times 10^9 \ (Hz).$$

$$\beta = \omega\sqrt{\mu\varepsilon} \cdot \sqrt{1-\left(\frac{f_c}{f}\right)^2} = 287.9 \ (rad/m).$$

$$\alpha_d = \frac{\sigma\eta}{2\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = 1.371 \times 10^{-8} \ (Np/m).$$

$$\alpha_c = \frac{2}{b\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \cdot \sqrt{\frac{\pi f \varepsilon}{\sigma_c}} = 4.530 \times 10^{-3} \ (Np/m).$$

$$u_p = u \cdot \frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = 2.182 \times 10^8 \ (m/s).$$

$$u_g = u \cdot \sqrt{1-\left(\frac{f_c}{f}\right)^2} = 1.833 \times 10^8 \ (m/s).$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = 2.182 \times 10^{-2} \ (m).$$

#9. (a). TE_1 $\quad f_c = (f_c)_{TM_1}$.

$$\alpha_c = \frac{2}{b\sqrt{1-\left(\frac{f_c}{f}\right)}} \cdot \sqrt{\frac{\pi f \varepsilon}{\sigma_c}} \cdot \left(\frac{f_c}{f}\right)^2 = 1.695 \times 10^{-4} \ (Np/m).$$

All others are the same as ~~TM~~ parameters of TM_1.

(b). TE_2 mode. $(f_c) = (f_c)_{TM_2} = 4 \times 10^9$ Hz

$$\alpha_c = \frac{2}{b\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \cdot \sqrt{\frac{\pi f \varepsilon}{\sigma_c}} \cdot \left(\frac{f_c}{f}\right)^2 = 7.249 \times 10^{-4} \ (Np/m).$$

All others are the same as parameters of TM_2 mode.