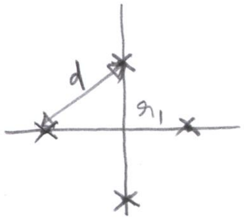
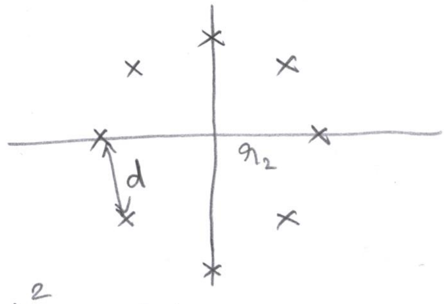
ECE 146 B
Gibson

Spring 2008
05 / 22 / 2008

Mid Term Solutions

1.





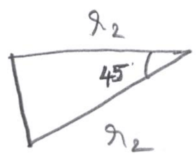
$$d = \sqrt{r_1^2 + r_1^2}$$

$$\boxed{r_1 = \frac{d}{\sqrt{2}}}$$

$$E_s = r_1^2$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$d^2 = r_2^2 + r_2^2 - 2 r_2 r_2 \cos 45^\circ$$



$$= r_2^2 (2 - \sqrt{2})$$

$$\boxed{r_2 = \frac{d}{\sqrt{2-\sqrt{2}}}}$$

$$E_s = r_2^2$$

$$P_{4PSK} = 10 \log_{10} \frac{r_1^2}{2T_b} \quad \text{dB}$$

$$P_{8PSK} = 10 \log_{10} \frac{r_2^2}{3T_b} \quad \text{dB}$$

T_b = bit duration

4 PSK : 2 bits per symbol

8 PSK : 3 bits per symbol

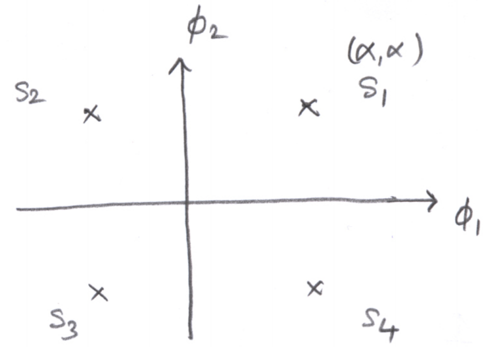
Additional power needed by 8 PSK :

$$P_{8PSK} - P_{4PSK} = 10 \log_{10}\left(\frac{d^2}{3T_b\,(2-\sqrt{2})}\right) - 10 \log_{10}\left(\frac{d^2}{2T_b\, 2}\right)$$

$$= 10 \log_{10}\left(\frac{4}{3(2-\sqrt{2})}\right)$$

$$= \underline{\underline{3.572 \text{ dB}}}$$

2. $\underline{y} = \underline{x} + \underline{n}$



where $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\underline{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$

$\underline{x}$ is one of $\underset{s_1}{\begin{bmatrix} \alpha \\ \alpha \end{bmatrix}}$, $\underset{s_4}{\begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}}$, $\underset{s_2}{\begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}}$, $\underset{s_3}{\begin{bmatrix} -\alpha \\ -\alpha \end{bmatrix}}$

$n_1 \sim N(0, \sigma^2)$, $n_2 \sim N(0, \sigma^2)$ are independent

The decision regions for the symbols are:

s_1 : $0 \le y_1 < \infty$, $0 \le y_2 < \infty$

s_2 : $-\infty < y_1 < 0$, $0 \le y_2 < \infty$

s_3 : $-\infty < y_1 < 0$, $-\infty < y_2 < 0$

s_4 : $0 \le y_1 < \infty$, $-\infty < y_2 < 0$

Conditioned on s_1 being transmitted, we have

$$y_1 = \alpha + n_1$$
$$y_2 = \alpha + n_2$$

$y_1 \sim N(\alpha, \sigma^2)$, $y_2 \sim N(\alpha, \sigma^2)$

Probability of no error is given by:

$$P_{c/s_1} = P(0 \le y_1 < \infty,\ 0 \le y_2 < \infty)$$

$$= P(0 \le y_1 < \infty)\ P(0 \le y_2 < \infty) \quad \text{(independent)}$$

$$= \left[\frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty e^{-\frac{(y_1-\alpha)^2}{2\sigma^2}} dy_1 \right]^2$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{-\frac{\alpha}{\sigma}}^\infty e^{-\frac{z^2}{2}} dz \right]^2 \qquad z = \frac{y_1 - \alpha}{\sigma}$$

$$= \left[1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{\alpha}{\sigma}} e^{-\frac{z^2}{2}} \, dz\right]^2$$

$$= \left[1 - \frac{1}{\sqrt{2\pi}} \int_{\frac{\alpha}{\sigma}}^{\infty} e^{-\frac{z^2}{2}} \, dz\right]^2$$

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{z^2}{2}} \, dz$$

$$= \left(1 - Q\left(\frac{\alpha}{\sigma}\right)\right)^2$$

$Z \sim N(0,1)$

$$= \left(1 - \frac{1}{2}\, \mathrm{erfc}\left(\frac{\alpha}{\sqrt{2}\sigma}\right)\right)^2.$$

$$Q(x) = \frac{1}{2}\, \mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

Probability of error:

$$P_{e/s_1} = 1 - P_{c/s_1} = 1 - \left[1 - \frac{1}{2}\, \mathrm{erfc}\left(\frac{\alpha}{\sqrt{2}\sigma}\right)\right]^2.$$

Similarly, it can be shown that the probability of error conditioned on each symbol sent is the same,

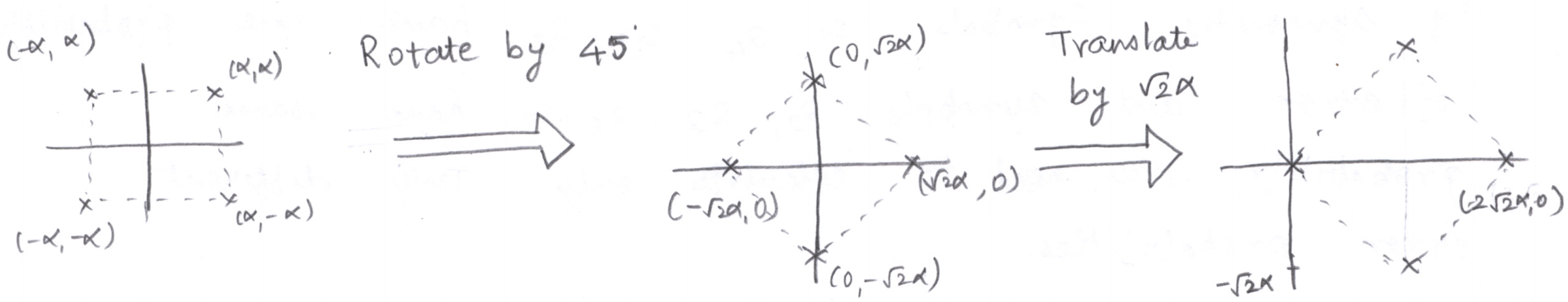
ie, $P_{e/s_2} = P_{e/s_3} = P_{e/s_4} = P_{e/s_1}$.

$\therefore$ Average probability of error:

$$P_e = \frac{1}{4} P_{e/s_1} + \frac{1}{4} P_{e/s_2} + \frac{1}{4} P_{e/s_3} + \frac{1}{4} P_{e/s_4}$$

(σ^2 is the noise variance)

$$= 1 - \left[1 - \frac{1}{2}\, \mathrm{erfc}\left(\frac{\alpha}{\sqrt{2}\sigma}\right)\right]^2 = \mathrm{erfc}\left(\frac{\alpha}{\sqrt{2}\sigma}\right) - \frac{1}{4}\left(\mathrm{erfc}\left(\frac{\alpha}{\sqrt{2}\sigma}\right)\right)^2.$$



Using the principles of rotational and translational invariance, we can conclude that the probability of error is the same.

3. Zero-forcing equalizer: $\sum_{n=-N}^{+N} w_k x_{n-k} = \begin{cases} 1 & n=0 \\ 0 & n = \pm 1, \pm 2, \dots, \pm N \end{cases}$

$x_0 = x(0) = 1, \quad x_{-1} = x(-1) = 0.3, \quad x_1 = x(1) = 0.2.$

$w_{-1} x_1 + w_0 x_0 + w_1 x_{-1} = 1 \Rightarrow 0.2 w_{-1} + w_0 + 0.3 w_1 = 1 \rightarrow (1)$

$w_0 x_1 + w_1 x_0 = 0 \Rightarrow 0.2 w_0 + w_1 = 0 \rightarrow (2)$

$w_{-1} x_0 + w_0 x_{-1} = 0 \Rightarrow w_{-1} + 0.3 w_0 = 0 \rightarrow (3)$

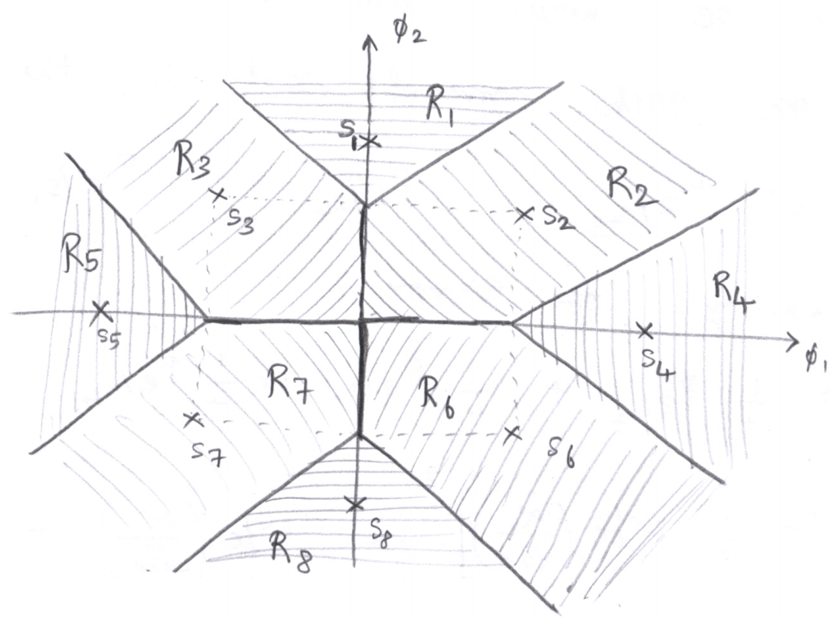
Solving for w_0, w_{-1}, w_1 in (1), (2) and (3), we get

$w_0 = 1.136$

$w_1 = -0.2272$

$w_{-1} = -0.3408.$

4.



By symmetry, symbols s_1, s_4, s_5, s_8 have same probability of error and symbols s_2, s_3, s_6, s_7 have same probability. We need to calculate only <u>TWO</u> different error probabilities.

$$P_e = \frac{1}{2} P_{e/s_1} + \frac{1}{2} P_{e/s_2}$$

5. $R_s = 60\,000$ symbols/sec, $\alpha = 0.5$

Bandwidth required:

$$BW = W(1+\alpha) = \frac{R_s}{2}(1+\alpha) = 30000\,(1.5) = \underline{\underline{45 \text{ KHz}}}$$

6. $$s_i(t) = \sqrt{E}\sqrt{\frac{2}{T}}\cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right]$$

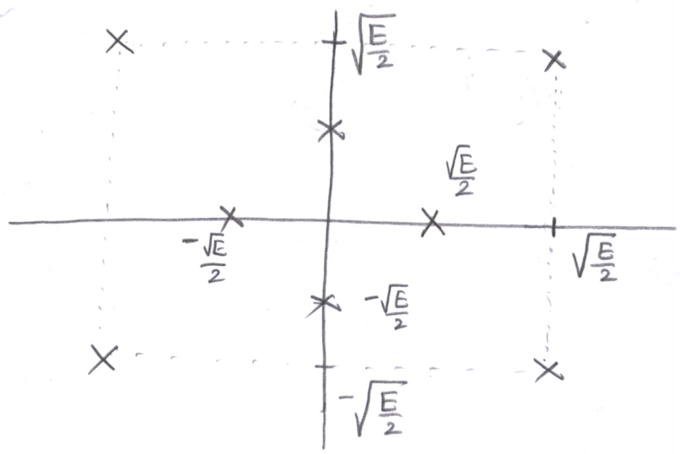
$$= \sqrt{E}\cos(2i-1)\frac{\pi}{4}\left(\sqrt{\frac{2}{T}}\cos 2\pi f_c t\right) - \sqrt{E}\sin(2i-1)\frac{\pi}{4}\left(\sqrt{\frac{2}{T}}\sin 2\pi f_c t\right)$$

$i = 1, 2, 3, 4$

$$s_j(t) = \frac{1}{2}\sqrt{\frac{2E}{T}}\cos\left[2\pi f_c t + (2j)\frac{\pi}{4}\right]$$

$$= \frac{\sqrt{E}}{2}\cos(2j)\frac{\pi}{4}\left(\sqrt{\frac{2}{T}}\cos 2\pi f_c t\right) - \frac{\sqrt{E}}{2}\sin(2j)\frac{\pi}{4}\left(\sqrt{\frac{2}{T}}\sin 2\pi f_c t\right)$$

$j = 0, 1, 2, 3$



7. $h(t) = s(T-t)$

Optimum sampling time : $t = T$

$$SNR_{opt} = \frac{2E}{N_0}$$

$$E = \int_0^T s^2(t)\,dt = \int_0^T \left(\frac{At}{T}\right)^2 dt = \frac{A^2 T}{3}.$$

$\frac{N_0}{2}$ is the noise power spectral density.

$$SNR_{opt} = \frac{2A^2T}{3N_0}$$