ECE 146B
Gibson

Solutions to Homework # 6

Spring 2008
06 / 03 /2008

1. Let $Z = X + iY$

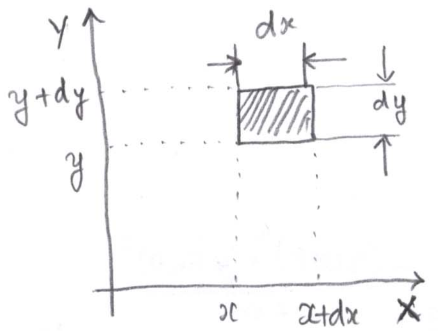
If X and Y are zero mean independent Gaussian random variables and equal variance, then $|Z| = \sqrt{X^2+Y^2}$, the envelope is a Rayleigh random variable.

Let $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0, \sigma^2)$.

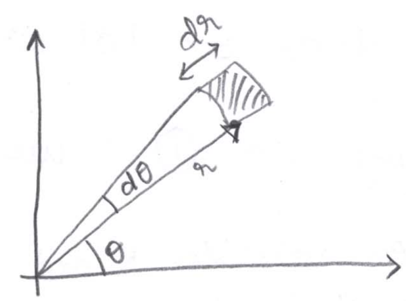
The joint pdf of X and Y is:

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



Cartesian co ordinates



Polar co ordinates

$$P[x \le X \le x+dx,\ y \le Y \le y+dy] = f_{XY}(x,y)\, dx\, dy \quad \text{(dx, dy are very small)}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} dx\, dy \longrightarrow (1)$$

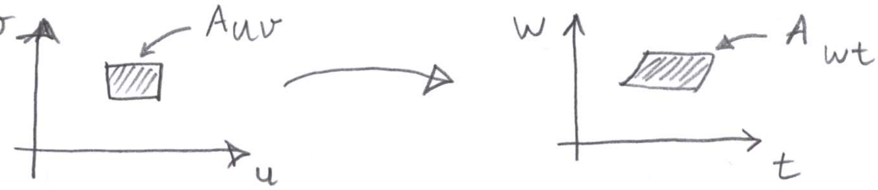
Coordinate transformation:

$$x = r\cos\theta \qquad\qquad r = \sqrt{x^2+y^2}$$
$$\Rightarrow$$
$$y = r\sin\theta, \qquad\qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

When we have a transformation of the form

$w = f(u,v)$

$t = g(u,v)$



the ratio of the elemental areas is given by

$$\frac{A_{wt}}{A_{uv}} = |J|$$ where J is called the Jacobian of the transformation and is defined as

$$J = \begin{vmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}$$ (determinant)

Applying this to the transformation $x = r\cos\theta$, $y = r\sin\theta$, we get:

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

$$A = dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta.$$

Substituting in ①, we get:

$$\Pr\left[\begin{array}{l}\text{Random variables } R \text{ and } \Theta \text{ lie} \\ \text{in the elemental area in the} \\ \text{polar coordinate system}\end{array}\right] = \frac{1}{2\pi\sigma^2} e^{-\frac{(r\cos\theta)^2 + (y\sin\theta)^2}{2\sigma^2}} r\,dr\,d\theta$$

$$f_{R,\theta}(r,\theta)\,dr\,d\theta = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \frac{1}{2\pi}\,dr\,d\theta$$

The joint pdf of R and θ is: $f_{R,\theta}(r,\theta) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \frac{1}{2\pi}$

The marginal pdf of R:

$$f_R(r) = \int_0^{2\pi} f_{R,\theta}(r,\theta)\,d\theta = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \sim \text{Rayleigh pdf}$$

Note that $f_\theta(\theta) = \frac{1}{2\pi}$, $0 \le \theta \le 2\pi$ $\sim$ uniform pdf.

2. $Z = X + iY$ where $X \sim N(A, \sigma^2) \Rightarrow$ non-zero mean

$Y \sim N(0, \sigma^2)$

X and Y are independent

$|Z| = \sqrt{X^2+Y^2}$ has a Rician distribution.

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-A)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{(x-A)^2+y^2}{2\sigma^2}}$$

Again applying the transformation $x = r\cos\theta$, $y = r\sin\theta$,

we get :

$$f_{R\theta}(r,\theta)\, dr\, d\theta = \frac{1}{2\pi\sigma^2} e^{-\frac{(r\cos\theta - A)^2 + r^2\sin^2\theta}{2\sigma^2}} r\, dr\, d\theta$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{r^2 - 2Ar\cos\theta + A^2}{2\sigma^2}} dr\, d\theta$$

$$\Rightarrow f_{R\theta}(r,\theta) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2+A^2 - 2Ar\cos\theta}{2\sigma^2}}$$

The marginal pdf of R:

$$f_R(r) = \int_0^{2\pi} f_{R\theta}(r,\theta)\, d\theta = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} \int_0^{2\pi} e^{\frac{Ar\cos\theta}{\sigma^2}} d\theta$$

$$= \frac{r}{\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} I_0\left(\frac{Ar}{\sigma^2}\right)$$

where $I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x\cos\theta} d\theta$

Problem 5.10

$$E[X_j W'(t_k)] = E[(s_{ij} + W_j)W'(t_k)]$$

$$E[s_{ij} \; W'(t_k)] = s_{ij} \; E[W'(t_k)] = 0$$

We also note that

$$W'(t_k) = W(t_k) - \sum_{i=1}^{N} W_i \; \phi_i(t_k)$$

We therefore have

$$E[X_j W'(t_k)] = E[W_j \; W'(t_k)]$$

$$= E[W_j W(t_k) - \sum_{i=1}^{N} \phi_i(t_k) \; E[W_j W_i]$$

But $E[W_j \; W(t_k)] = E[W(t_k) \int_0^T W(t) \; \phi_j(t) dt] = \int_0^T \phi_j(t) \; E[W(t_k)W(t)]dt$

$$= \int_0^T \phi_j(t) \cdot \frac{N_0}{2} \delta(t-t_k)dt = \frac{N_0}{2} \phi_j(t_k)$$

$$E[W_j W_i] = \begin{cases} \frac{N_0}{2}, & i=j \\ 0 & i \neq j \end{cases}$$

Hence, we get the final result

$$E[X_j \; W'(t_k)] = \frac{N_0}{2} \phi_j(t_k) - \frac{N_0}{2} \phi_j(t_k)$$

$$= 0.$$

4. The symbol $\|\cdot\|$ is called the "norm" of a vector. For a vector $\underline{z} \in \mathbb{R}^N$, ie, $\underline{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$,

$$\|\underline{z}\| = \sqrt{z_1^2 + z_2^2 + \dots + z_n^2}$$

It is also the length of the vector.

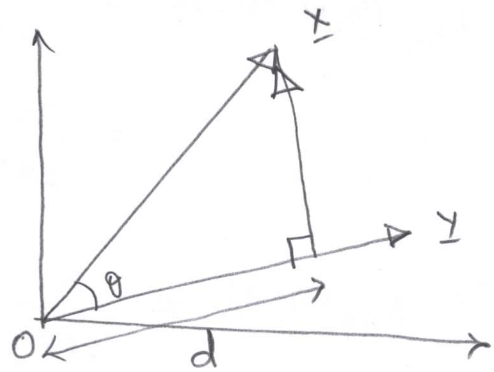
Also note that, $\underline{z}^T\underline{z} = [z_1 \; z_2 \cdots z_N] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$

$$= z_1^2 + z_2^2 + \dots + z_N^2 = \|\underline{z}\|^2$$

Now let us take two vectors $\underline{x}$ and $\underline{y}$ in $\mathbb{R}^2$ (2D space)

Let θ be the angle between the vectors

Draw a perpendicular from $\underline{x}$ to $\underline{y}$.



Now, $\cos\theta = \dfrac{d}{\text{length of } \underline{x}} = \dfrac{d}{\|\underline{x}\|}$.

The perpendicular line itself is a vector and it can be written as: $\underline{r} = \underline{x} - d \dfrac{\underline{y}}{\|\underline{y}\|}$

where $\dfrac{\underline{y}}{\|\underline{y}\|}$ is the unit vector in the direction of the vector $\underline{y}$. Since $\underline{r}$ and $\underline{y}$ are orthogonal,

$\underline{r}^T\underline{y} = 0$ (inner product or dot product is zero)

$$\left(\underline{x} - d\,\frac{\underline{y}}{\|\underline{y}\|}\right)^T \underline{y} = 0$$

(d and $\|\underline{y}\|$ are scalars)

$$\underline{x}^T\underline{y} = d\,\frac{\underline{y}^T\underline{y}}{\|\underline{y}\|}$$

$$\underline{x}^T\underline{y} = d\,\frac{\|\underline{y}\|^2}{\|\underline{y}\|} \Rightarrow d = \frac{\underline{x}^T\underline{y}}{\|\underline{y}\|}$$

$$\therefore \cos\theta = \frac{d}{\|\underline{x}\|} = \frac{\underline{x}^T\underline{y}}{\|\underline{x}\|\,\|\underline{y}\|}$$

Since $|\cos\theta| \le 1$, $\dfrac{|\underline{x}^T\underline{y}|}{\|\underline{x}\|\,\|\underline{y}\|} \le 1$

$$\Rightarrow |\underline{x}^T\underline{y}| \le \|\underline{x}\|\,\|\underline{y}\|$$

In the signal space, we represent signals by vectors and the energy of the signal is the length squared of the corresponding vector. The inner product (correlation) between real signals $x(t)$ and $y(t)$ is $\int_{-\infty}^{\infty} x(t)\,y(t)\,dt$.

$\therefore$ we have

$$\left|\int_{-\infty}^{\infty} x(t)\,y(t)\,dt\right| \le \sqrt{\int_{-\infty}^{\infty} x^2(t)\,dt}\ \sqrt{\int_{-\infty}^{\infty} y^2(t)\,dt}$$

For complex signals,

$$\left|\int_{-\infty}^{\infty} x(t)\,y^*(t)\,dt\right| \le \sqrt{\int_{-\infty}^{\infty} |x^2(t)|\,dt}\ \sqrt{\int_{-\infty}^{\infty} |y^2(t)|\,dt}$$

5. $$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \longrightarrow (1)$$

$$\phi_2(t) = \frac{s_2(t) - s_{21}\,\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$ where $s_{21} = \int_0^T s_2(t)\,\phi_1(t)\,dt \longrightarrow (2)$

$$\int_0^T \phi_1(t)\,\phi_2(t)\,dt = \int_0^T \phi_1(t)\left[\frac{s_2(t) - s_{21}\,\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}\right]dt$$

$$= \frac{1}{\sqrt{E_2 - s_{21}^2}}\int_0^T s_2(t)\,\phi_1(t) - \frac{s_{21}}{\sqrt{E_2 - s_{21}^2}}\int_0^T \underbrace{\phi_1^2(t)\,dt}_{=1}$$

$\phi_1(t)$ has unit energy

$$= \frac{s_{21}}{\sqrt{E_2 - s_{21}^2}} - \frac{s_{21}}{\sqrt{E_2 - s_{21}^2}} = 0$$

from (2)

$\Rightarrow$ $\phi_1(t)$ and $\phi_2(t)$ are orthonormal

Note that $\phi_2(t)$ also has unit energy.