ECE 146B
Gibson

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Solutions to Homework No. 4

1. $Y = aX + b = g(x) \Rightarrow x = g^{-1}(y) = \frac{y-b}{a}$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|} = \frac{f_X\left(\frac{y-b}{a}\right)}{|a|}$$

$$= \frac{1}{|a|}\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{y-b}{a}-\mu\right)^2/2\sigma^2} = \frac{1}{\sqrt{2\pi(a\sigma)^2}} e^{-\left[y-(a\mu+b)\right]^2/2(a\sigma)^2}$$

$\therefore$ Y is Gaussian with mean $= a\mu + b$ and variance $= a^2\sigma^2$.

2. $Y = X_1 + X_2 + \dots + X_p$

Since X_i are iid, $f_Y(y) = f_{X_1}(x_1) * f_{X_2}(x_2) * \dots\dots * f_{X_p}(x_p)$

We can make use of the moment generating function, that will enable use ~~to~~ do multiplication instead of convolution.

Consider $Z = X_1 + X_2$

$$E[e^{tZ}] = \int_{-\infty}^{\infty} e^{tz} f_Z(z)\,dz = \int_{-\infty}^{\infty} e^{tz}\left(f_{X_1}(x_1) * f_{X_2}(x_2)\right) dz$$

$$= \int_{-\infty}^{\infty} e^{tz} \int_{-\infty}^{\infty} f_{X_1}(u)\, f_{X_2}(z-u)\,du\,dz$$

Substituting $m = z - u$ and rearranging gives the result,

$$E[e^{tZ}] = E[e^{tX_1}]\, E[e^{tX_2}].$$

Similarly extending for p random variables $X_1, \dots, X_p$, we get,

$$E[e^{tY}] = E[e^{tX_1}]\, E[e^{tX_2}] \cdots\cdots E[e^{tX_p}] \longrightarrow (1)$$

Let us find the moment generating function of a Gaussian random variable with mean μ and variance σ^2

$$E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[(x-\mu)^2 - 2\sigma^2 tx\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2(\sigma^2 t + \mu)x + \mu^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2(\sigma^2 t + \mu)x + \underbrace{\mu^2 + 2\mu\sigma^2 t + (\sigma^2 t)^2}_{(\mu+\sigma^2 t)^2} \underbrace{- 2\mu\sigma^2 t - (\sigma^2 t)^2}_{\text{constants}}\right]} dx$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x - (\mu+\sigma^2 t))^2} dx}$$

$\longrightarrow$ Gaussian pdf with mean $= \mu + \sigma^2 t$ and variance σ^2

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} (1) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

Substitute in (1),

$$E[e^{tY}] = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \cdot e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} \cdots\cdots e^{\mu_p t + \frac{\sigma_p^2 t^2}{2}}$$

Here $\mu_1 = \mu_2 = \dots = \mu_p = \mu = 1$ and $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_p^2 = 2$

$$E[e^{tY}] = e^{P\mu + P\sigma^2\frac{t^2}{2}}$$

Note that this is the moment generating function of a Gaussian random variable with mean $= P\mu$ and variance $= P\sigma^2$

$\therefore$ Y is Gaussian. $f_Y(y) = \frac{1}{\sqrt{2\pi P\sigma^2}} e^{-(x-P\mu)^2/2P\sigma^2}$.

Alternate method:

Consider $Z = X_1 + X_2$

$$f_Z(z) = f_{X_1}(x_1) * f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1}(u) f_{X_2}(z-u)\,du$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(u-\mu)^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-u-\mu)^2/2\sigma^2}\,du$$

$(\because X_1 \sim N(\mu, \sigma^2)$ and $X_2 \sim N(\mu, \sigma^2))$

$$= \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[u^2 - 2\mu u + \mu^2 + z^2 - 2zu + u^2 - 2z\mu + 2\mu u + \mu^2\right]}\,du$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[2u^2 - 2zu + z^2 + \underbrace{2\mu^2 - 2z\mu}_{\text{constant}}\right]}\,du$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left[2\mu^2 - 2z\mu\right]}\int_{-\infty}^{\infty} e^{-\frac{1}{2\frac{\sigma^2}{2}}\left[u^2 - zu + \frac{z^2}{2}\right]}\,du$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left[2\mu^2 - 2z\mu\right]} \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\frac{\sigma^2}{2}}\left[u^2 - 2u\frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^2\right]}\,du$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left[2\mu^2 - 2z\mu + 2\left(\frac{z}{2}\right)^2\right]} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2\pi}\frac{\sigma}{\sqrt{2}}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\left(\frac{\sigma}{\sqrt{2}}\right)^2}\left[u - \frac{z}{2}\right]^2}\,du\right)$$

the term in brackets is a Gaussian pdf with mean $= \frac{z}{2}$ and variance $= \frac{\sigma^2}{2}$, integrated from $-\infty$ to ∞

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}\left[2\mu^2 - 2z\mu + \frac{z^2}{2}\right]} \quad (1)$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} e^{-\frac{1}{2\cdot 2\sigma^2}\left[z^2 - 2z(2\mu) + 4\mu^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} e^{-\frac{1}{2(\sqrt{2}\sigma)^2}\left[z - 2\mu\right]^2}$$

$\therefore z$ is Gaussian with mean 2μ and variance $2\sigma^2$

Now consider $Z_1 = X_1 + X_2 + X_3 = Z_1 + X_3$ where $Z_1 \sim \mathcal{N}(2\mu, 2\sigma^2)$ and $X_3 \sim N(\mu, \sigma^2)$

Using similar argument, we can show that

Z_1 is Gaussian with mean 3μ and variance $3\sigma^2$

Extending the argument to P variables,

$Y = X_1 + X_2 + \dots + X_P$ has a Gaussian pdf

with mean $= P\mu$ and variance $= P\sigma^2$.

3. (a) WSS : We need $\mu_Z(t) =$ constant and $R_Z(t, t+\tau) = R_Z(\tau)$

$Z(t) = X\cos\omega_c t + Y\sin\omega_c t$

$E[Z(t)] = \mu_Z(t) = \cos\omega_c t\ \mu_X + \sin\omega_c t\ \mu_Y$

$\mu_Z(t) =$ constant only when $\mu_X = \mu_Y = 0 \rightarrow (1)$

$R_Z(t, t+\tau) = E[Z(t)\, Z(t+\tau)]$

$= E\left[(X \cos w_c t + Y \sin w_c t)\, (X \cos w_c (t+\tau) + Y \sin w_c (t+\tau)) \right]$

$= E[X^2 \cos w_c t \cos w_c (t+\tau)] + E[Y^2 \sin w_c t \sin w_c (t+\tau)]$

$+ E[XY] \cos w_c t \sin w_c (t+\tau) + E[XY] \sin w_c t \cos w_c (t+\tau)$

$E[XY] = E[X]E[Y] = \mu_X \mu_Y$ (since X and Y are independent)

We also need $\mu_X = \mu_Y = 0$ (from ①)

$R_Z(t, t+\tau) = E[X^2] \cos w_c t \cos w_c (t+\tau) + E[Y^2] \sin w_c t \sin w_c (t+\tau)$

If $E[X^2] = E[Y^2]$, $R_Z(t, t+\tau) = E[X^2] \cos w_c (t+\tau - t)$

$= E[X^2] \cos w_c \tau = R_Z(\tau)$

depends only on τ.

$\therefore Z(t)$ is WSS if $\mu_X = \mu_Y = 0$ and $\sigma_X^2 = \sigma_Y^2$.

(b) We already showed in Problem 2 that sum of independent Gaussian random variables is again a Gaussian. This is true for sum of scaled Gaussians also, except that the means and variances are scaled appropriately.

$Z(t) = X \cos w_c t + Y \sin w_c t$

$\mu_Z(t) = \mu_X \cos w_c t + \mu_Y \sin w_c t$

$\sigma_Z^2(t) = \cos^2 w_c t\, \sigma_X^2 + \sin^2 w_c t\, \sigma_Y^2$

$$f_Z(z;t) = \frac{1}{\sqrt{2\pi}\, \sigma_Z(t)}\, e^{-(z - \mu_Z(t))^2 / 2\sigma_Z^2(t)}$$

(c) In part (a), we never made use of the Gaussian assumption to derive conditions for wide sense stationarity. The conditions were independent of the pdf of X and Y. Hence, the Gaussian assumption is not needed for WSS.

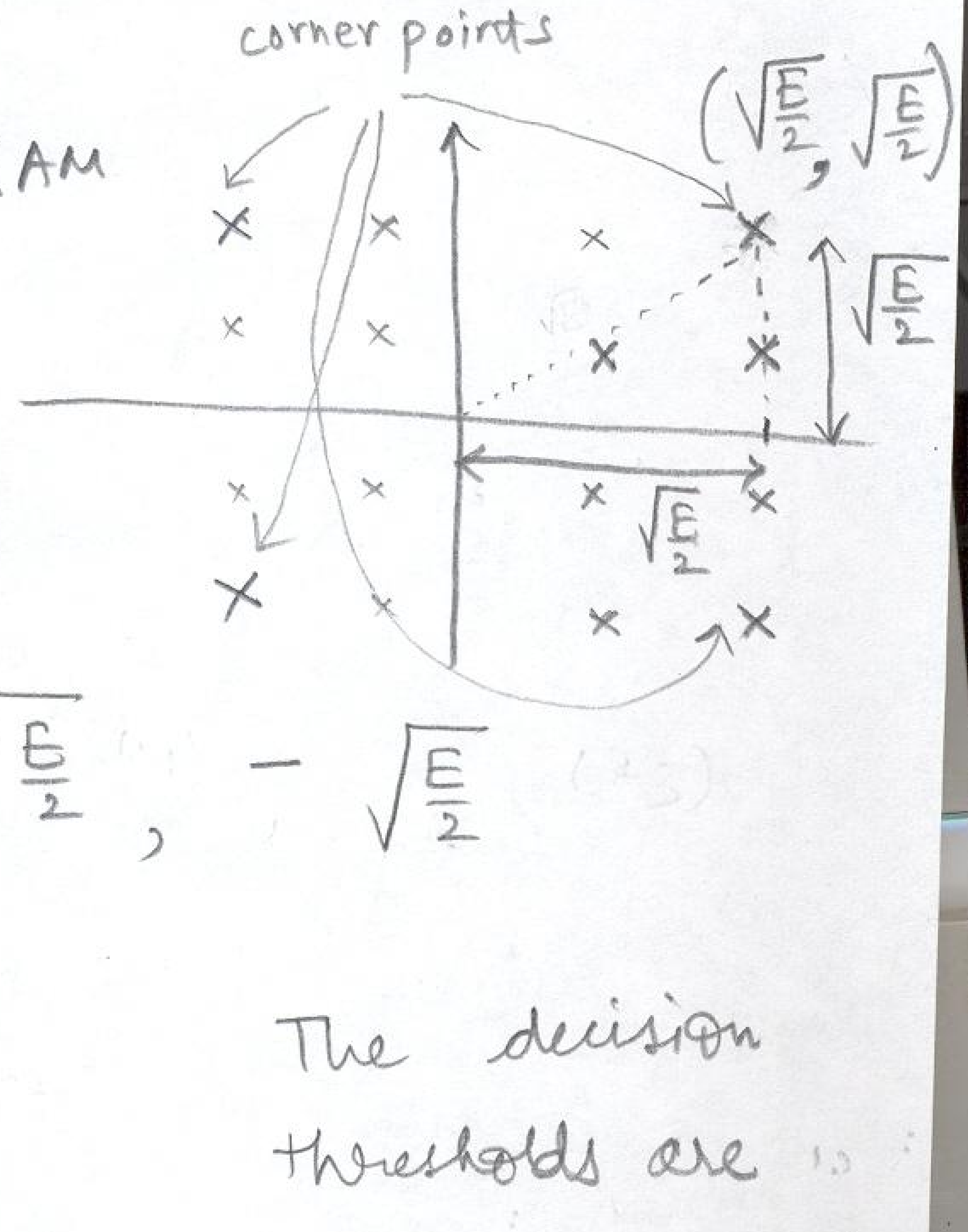
4. M QAM $\equiv$ 2 independent $\sqrt{M}$ PAMs in I and Q.

A symbol error can be caused by an error in the I direction or an error in the Q direction.

$$\therefore P_{e_{M\,QAM}} = 1 - P_{correct}$$

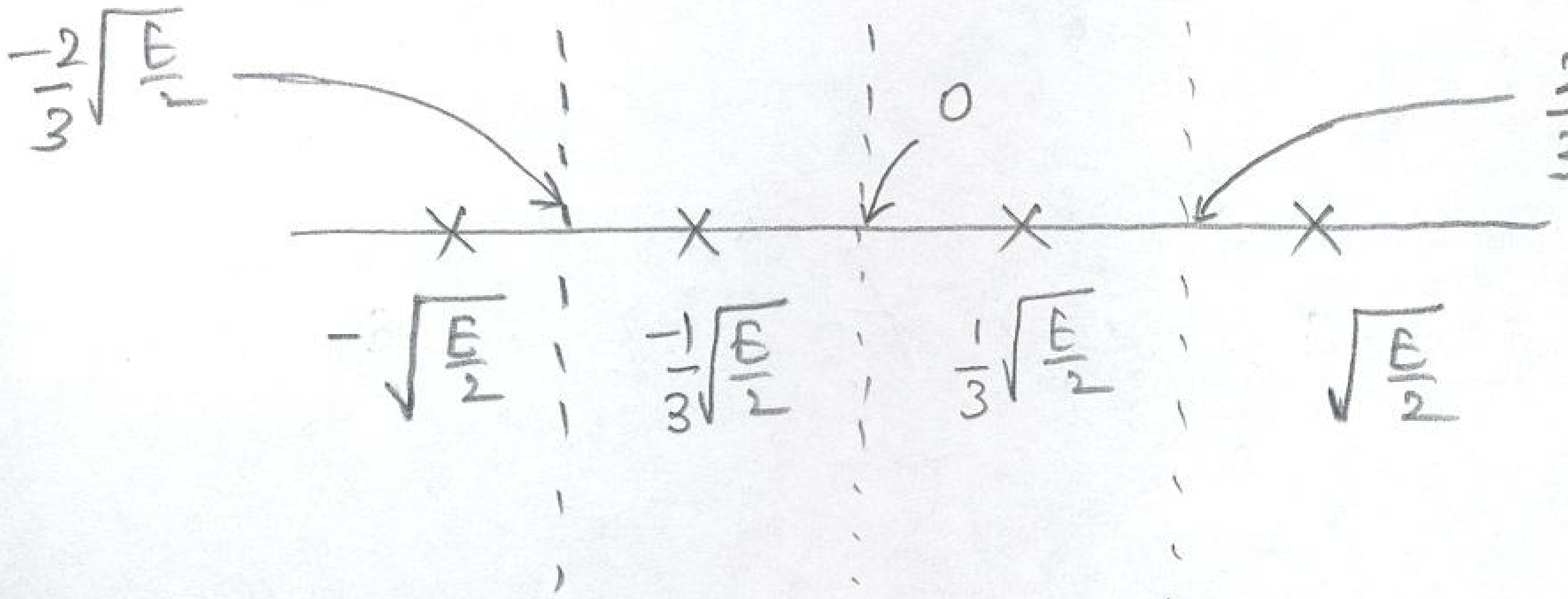
$$= 1 - \underbrace{(1 - P_{e_{\sqrt{M}\,PAM}})}_{\text{prob. of correct decoding in I}} \underbrace{(1 - P_{e_{\sqrt{M}\,PAM}})}_{\text{prob. of correct decoding in Q}}$$

Here, $M = 16$, $\sqrt{M} = 4$.



Also, the corner points in the 16 QAM constellation have energy E.

$\therefore$ The 4 points in the 4-PAM constellation are $+\sqrt{\frac{E}{2}}$, $\frac{1}{3}\sqrt{\frac{E}{2}}$, $-\frac{1}{3}\sqrt{\frac{E}{2}}$, $-\sqrt{\frac{E}{2}}$,



The decision thresholds are at $-\frac{2}{3}\sqrt{\frac{E}{2}}$, 0, $\frac{2}{3}\sqrt{\frac{E}{2}}$

We will evaluate P_e 4 PAM.

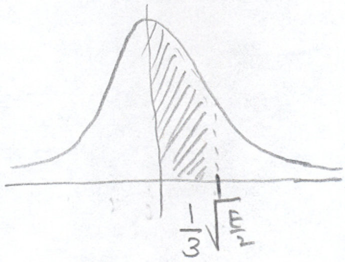
Let n be the additive white Gaussian noise with zero mean and variance $\frac{N_0}{2}$.

Then, probability of error for the symbols at $\pm\frac{1}{3}\sqrt{\frac{E}{2}}$ is:

$$P_e = 1 - P_{correct}$$

$$= 1 - P\left[-\frac{1}{3}\sqrt{\frac{E}{2}} \le n \le \frac{1}{3}\sqrt{\frac{E}{2}}\right] = 1 - 2P\left[0 \le n \le \sqrt{\frac{E}{2}}\right]$$

$$= 1 - 2\left[\frac{1}{2} - P\left[\frac{1}{3}\sqrt{\frac{E}{2}} \le n \le \infty\right]\right]$$



$$= 2\frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0}{2}}}\int_{\frac{1}{3}\sqrt{\frac{E}{2}}}^{\infty} e^{-\frac{n^2}{2\frac{N_0}{2}}}\,dn$$

Put $t = \frac{n}{\sqrt{\frac{N_0}{2}}}$

$dn = \sqrt{\frac{N_0}{2}}\,dt$

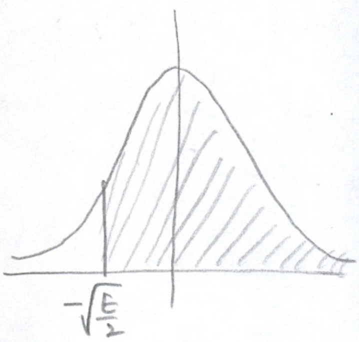
$$= 2\frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0}{2}}}\int_{\frac{1}{3}\sqrt{\frac{E}{N_0}}}^{\infty} e^{-\frac{t^2}{2}}\sqrt{\frac{N_0}{2}}\,dt$$

$$= 2\frac{1}{\sqrt{2\pi}}\int_{\frac{1}{3}\sqrt{\frac{E}{N_0}}}^{\infty} e^{-t^2/2}\,dt = 2Q\left(\frac{1}{3}\sqrt{\frac{E}{N_0}}\right)$$ where $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-t^2/2}\,dt$

For the symbols at the edges, there is a threshold in only direction. We make an error only when the noise pushes the received symbol beyond the threshold.
The prob. of error should be reduced by half to $Q\left(\sqrt{\frac{E}{N_0}}\right)$

For example, for the symbol at $\sqrt{\frac{E}{2}}$,

$$P_e = 1 - P\left[-\frac{1}{3}\sqrt{\frac{E}{2}} \le n \le \infty\right] = 1 - \frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0}{2}}}\int_{-\frac{1}{3}\sqrt{\frac{E}{2}}}^{\infty} e^{-\frac{n^2}{2\frac{N_0}{2}}}\,dn$$



$$= 1 - Q\left(-\frac{1}{3}\sqrt{\frac{E}{N_0}}\right) \qquad \left[Q(-x) = 1 - Q(x)\right]$$

$$= 1 - \left(1 - Q\left(\frac{1}{3}\sqrt{\frac{E}{N_0}}\right)\right)$$

$$= Q\left(\frac{1}{3}\sqrt{\frac{E}{N_0}}\right).$$

Now,

$$P_{e_{4PAM}} = \frac{\overbrace{4-2}^{M-2 \text{ inner symbols}}}{4} 2Q\left(\frac{1}{3}\sqrt{\frac{E}{N_0}}\right) + \overbrace{\frac{2}{4}}^{2 \text{ outer symbols}} Q\left(\frac{1}{3}\sqrt{\frac{E}{N_0}}\right).$$

$$= \frac{3}{2} Q\left(\frac{1}{3}\sqrt{\frac{E}{N_0}}\right).$$

$$\therefore P_{e_{16QAM}} = 1 - \left(1 - \frac{3}{2} Q\left(\frac{1}{3}\sqrt{\frac{E}{N_0}}\right)\right)\left(1 - \frac{3}{2} Q\left(\frac{1}{3}\sqrt{\frac{E}{N_0}}\right)\right)$$

$$= 3 Q\left(\frac{1}{3}\sqrt{\frac{E}{N_0}}\right) - \frac{9}{4}\left(Q\left(\frac{1}{3}\sqrt{\frac{E}{N_0}}\right)\right)^2$$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \quad \text{and} \quad Q(x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right).$$

In terms of $\operatorname{erfc}(x)$,

$$P_{e_{16QAM}} = \frac{3}{2}\operatorname{erfc}\left(\frac{1}{3}\sqrt{\frac{E}{2N_0}}\right) - \frac{9}{16}\left(\operatorname{erfc}\left(\frac{1}{3}\sqrt{\frac{E}{2N_0}}\right)\right)^2.$$