ECE 146B | | Spring 2009
Gibson | Homework No. 1 | Due: April 7, 2009

1. Given a Gaussian random variable X with mean 1 and variance 4, find the probability that X is greater than 5.

2. Problem 7.2 in the text.

3. Problem 7.4 in the text.

4. The attached derivation is from Appendix A of the Second edition of my book *Principles of Digital and Analog Communications,* Macmillan/Prentice-Hall, 1993. Note that the first and second editions are quite different. The problem assignment is: Write out this derivation in the notation of the current textbook, verifying each step in the process. This result will be used repeatedly in the course.

A.10 Cyclostationary Processes

A common model of transmitted sequences in digital communications systems is given by

$$X(t) = \sum_{n=-\infty}^{\infty} a_n p(t - nT_s), \tag{A.10.1}$$

where $p(t)$ is the pulse shape, T_s is the symbol duration, and $\{a_n\}$ is a WSS sequence with $E\{a_n\} = \mu_a$ and $E\{a_n a_m\} = E\{a_l a_{l+k}\} = R_a(k)$, $k = |n - m|$. We would like to find the power spectral density of $X(t)$. The mean of $X(t)$ is immediately available as

$$E[X(t)] = \mu_a \sum_{n=-\infty}^{\infty} p(t - nT_s), \tag{A.10.2}$$

and the autocorrelation is given by

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[a_n a_m] p(t_1 - nT_s) p(t_2 - mT_s) \\ &= \sum_{k=-\infty}^{\infty} R_a(k) \sum_{n=-\infty}^{\infty} p(t_1 - nT_s) p(t_2 - (k + n)T_s). \end{aligned} \tag{A.10.3}$$

From Eqs. (A.10.2) and (A.10.3) it is clear that the sequence $X(t)$ is not WSS. As a result, the power spectral density cannot be defined using Eq. (A.9.1).

Random processes that satisfy the relations

$$E[Y(t_1 + T)] = E[Y(t_1)] \tag{A.10.4}$$

and

$$R_Y(t_1 + T, t_2 + T) = R_Y(t_1, t_2) \tag{A.10.5}$$

are called *cyclostationary* because they are periodic in their time arguments [Franks, 1969]. We see from Eqs. (A.10.2) and (A.10.3) that the sequence $X(t)$ is a cyclostationary process. Fortunately, $X(t)$ can be modified to obtain a WSS process by allowing a random time delay.

Consider a new sequence

$$X(t) = \sum_{n=-\infty}^{\infty} a_n p(t - nT_s - \lambda), \tag{A.10.6}$$

where λ is a uniformly distributed random variable over $0 \le t < T_s$ independent of a_n. Then

$$
\begin{aligned}
E[X(t)] &= \sum_{n=-\infty}^{\infty} \mu_a E[p(t - nT_s - \lambda)] \\
&= \mu_a \sum_{n=-\infty}^{\infty} \frac{1}{T_s} \int_0^{T_s} p(t - nT_s - \lambda)\, d\lambda \\
&= \frac{\mu_a}{T_s} \sum_{n=-\infty}^{\infty} \int_{t-(n+1)T_s}^{t-nT_s} p(\alpha)\, d\alpha = \frac{\mu_a}{T_s} \int_{-\infty}^{\infty} p(t)\, dt, \qquad \text{(A.10.7)}
\end{aligned}
$$

which is a constant. Further,

$$
\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[a_n a_m] \int_0^{T_s} \frac{1}{T_s} p(t_1 - nT_s - \lambda) p(t_2 - mT_s - \lambda)\, d\lambda \\
&= \sum_{k=-\infty}^{\infty} R_a(k) \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \int_0^{T_s} p(t_1 - nT_s - \lambda) p(t_2 - (n+k)T_s - \lambda)\, d\lambda \\
&= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} R_a(k) \sum_{n=-\infty}^{\infty} \int_{t_1-(n+1)T_s}^{t_1-nT_s} p(\alpha) p(\alpha + \tau - kT_s)\, d\alpha \\
&= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} R_a(k) \int_{-\infty}^{\infty} p(t) p(t + \tau - kT_s)\, dt \\
&= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} R_a(k) \mathscr{R}_p(\tau - kT_s), \qquad \text{(A.10.8)}
\end{aligned}
$$

where $\tau = |t_2 - t_1|$ and

$$\mathscr{R}_p(\tau) = \int_{-\infty}^{\infty} p(t) p(t + \tau)\, dt. \qquad \text{(A.10.9)}$$

Since $R_X(t_1, t_2) = R_X(|t_2 - t_1|)$ and $E[X(t)] = \text{constant}$, $X(t)$ in Eq. (A.10.6) is WSS.

To simplify Eq. (A.10.8) further, assume that the a_n sequence is statistically independent (but not zero mean),

$$R_a(k) = E[a_n a_{n+k}] = \begin{cases} \mu_a^2, & k \neq 0 \\ \sigma_a^2 + \mu_a^2, & k = 0, \end{cases} \qquad \text{(A.10.10)}$$

where $\sigma_a^2 = E[a_n^2] - \mu_a^2$. Then, Eq. (A.10.8) yields

$$R_X(\tau) = \frac{\sigma_a^2}{T_s} \mathscr{R}_p(\tau) + \frac{\mu_a^2}{T_s} \sum_{k=-\infty}^{\infty} \mathscr{R}_p(\tau - kT_s). \qquad \text{(A.10.11)}$$

Using Eq. (A.10.9),

$$S_p(\omega) = \mathscr{F}\{\mathscr{R}_p(\tau)\} = |P(\omega)|^2, \qquad \text{(A.10.12)}$$

where $P(\omega) = \mathscr{F}\{p(t)\}$, we can write several different useful expressions for the power spectral density. Taking the Fourier transform of Eq. (A.10.8), we get the general relationship

$$S_X(\omega) = \frac{1}{T_s}|P(\omega)|^2 R_a(0) + \frac{1}{T_s}\sum_{k=-\infty}^{\infty} R_a(k)|P(\omega)|^2 e^{-j\omega k T_s}$$

$$= \frac{|P(\omega)|^2}{T_s}\left\{R_a(0) + 2\sum_{k=1}^{\infty} R_a(k)\cos k\omega T_s\right\}. \qquad \text{(A.10.13)}$$

Based on the assumptions in Eq. (A.10.10), we can start with Eq. (A.10.11) rewritten as

$$R_X(\tau) = \frac{\sigma_a^2}{T_s}\mathscr{R}_p(\tau) + \frac{\mu_a^2}{T_s^2}\sum_{k=-\infty}^{\infty} \mathscr{R}_p(\tau) * \delta(\tau - kT_s) \qquad \text{(A.10.14)}$$

and take the Fourier transform to get

$$S_X(\omega) = \frac{\sigma_a^2}{T_s}|P(\omega)|^2 + \frac{2\pi\mu_a^2}{T_s^2}\sum_{k=-\infty}^{\infty}\left|P\left(\frac{2k\pi}{T_s}\right)\right|^2 \delta\left(\omega - \frac{2k\pi}{T_s}\right). \qquad \text{(A.10.15)}$$

Equations (A.10.13) and (A.10.15) find application in several chapters of the book.