Homework-1 Solutions.

1. $P(X>5) = P\left(\frac{X-1}{2} > 2\right) = P(Y>2)$

$Y = N(0,1)$.

$= Q(2)$.

2. (a) For no aliasing to occur we must sample at the Nyquist rate

$f_s = 2 \times 6000$ samples/sec $= 12000$ samples/sec.

(b) With guard band of 2000,

$f_s - 2W = 2000 \Rightarrow f_s = \underline{14000}$

The reconstruction filter should not pick-up frequencies of images of spectrum $X(f)$. The nearest image spectrum is centered at f_s and occupies the frequency band $[f_s - W, f_s + W]$. Thus the highest frequency of the reconstruction filter ($= 10000$) should satisfy.

$10000 < f_s - W \Rightarrow f_s \geq 16000$

For value $f_s = 16000$, K should be such that

$K \cdot f_s = 1 \Rightarrow K = \frac{1}{16000}$

②

3. $x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, p(t-nT_s)$

$$= p(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t-nT_s)$$

$$= p(t) * x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

$$X_p(f) = P(f)\, X(f) * \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\right]$$

$$= P(f) \cdot X(f) * \left(\frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta\left(f-\frac{n}{T_s}\right)\right)$$

$$= \frac{1}{T_s} P(f) \sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{T_s}\right)$$

② $\frac{1}{T_s} > 2W$

③ $X(f)$ can be recovered using reconstruction filter $\Pi\left(\frac{f}{2W_r}\right)$ with $W < W_r < \frac{1}{T_s} - W$. In this case

$$X(f) = \frac{X_p(f)\, T_s\, \Pi\left(\frac{f}{2W_r}\right)}{P(f)}.$$

4. $$v(t) = \sum_{n=-\infty}^{\infty} I_n g(t-nT)$$

$R =$ bit rate

$\frac{1}{T} = R/k \Rightarrow$ symbol rate

$\{I_n\}$ = sequence of symbols.

$$\phi_{vv}(t+\tau, t) = E[v^*(t)\, v(t+\tau)]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[I_n^* I_m]\, g^*(t-nT)\, g(t+\tau-mT)$$

$\{I_n\}$ is WSS mean μ_i and autocorrelation $\phi_{ii}(m)$

$$\phi_{ii}(m) = E[I_n^* I_{n+m}]$$

$$\therefore \phi_{vv}(t+\tau, t) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} g^*(t-nT)\, g(t+\tau-nT-mT)$$

Now

$$\sum_{n=-\infty}^{\infty} g^*(t-nT)\, g(t+\tau-nT-mT)$$

is periodic in t with period T,

$\therefore \phi_{vv}(t+\tau, t)$ is also periodic in t with period T

$$\phi_{vv}(t+T+\tau; t+T) = \phi_{vv}(t+\tau; t)$$

also,

$$E[v(t)] = \mu_i \sum_{n=-\infty}^{\infty} g(t-nT)$$

$\therefore$ $v(t)$ is a stochastic process having periodic mean and autocorrelation function. Such a process is called cyclostationary process or periodically stationary process in the wide sense.

In order to compute PSD of cyclostationary process dependence of $\phi_{vv}(t+\tau; t)$ on variable t must be eliminated. This is accomplished by averaging $\phi_{vv}(t+\tau; t)$ over a single period. Thus,

$$\bar{\phi}_{vv}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} \phi_{vv}(t+\tau, t)\, dt$$

$$= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-T/2}^{T/2} g^*(t-nT)\, g(t+\tau-nT-mT)\, dt$$

$$= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-T/2-nT}^{T/2-nT} g^*(t)\, g(t+\tau-mT)\, dt$$

Lets define $\phi_{gg}(t) = \int_{-\infty}^{\infty} g(t)\, g^*(t+\tau)\, dt$

⑤

$$\phi_{vv}(\tau) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \phi_{ii}(m)\, \phi_{gg}(\tau - mT)$$

$$\Phi_{vv}(f) = \frac{1}{T} |G(f)|^2 \Phi_{ii}(f)$$

$$\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m) e^{-j2\pi f mT}$$

<u>Now</u>

$$\phi_{ii}(m) = T \int_{-1/2T}^{1/2T} \Phi_{ii}(f) e^{j2\pi f mT} df$$

because $\Phi_{ii}(f)$ is periodic with period $1/T$ as $\phi_{ii}(\tau)$ is periodic with period T.

Now for special case where $\{I_n\}$ are real and mutually uncorrelated,

$$\phi_{ii}(m) = \begin{cases} \sigma_i^2 + \mu_i^2 & (m = 0) \\ \mu_i^2 & (m \neq 0) \end{cases}$$

$$\Phi_{ii}(f) = \sigma_i^2 + \mu_i^2 \underbrace{\sum_{m=-\infty}^{\infty} e^{-j2\pi f mT}}$$

$$\downarrow$$

$$\frac{1}{T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right)$$

$$\therefore \quad \Phi_{ii}(f) = \sigma_i^2 + \frac{\mu_i^2}{T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right)$$

$$\therefore \quad \Phi_{vv}(f) = \frac{\sigma_I^2}{T} |G(f)|^2 + \frac{\mu_i^2}{T^2} \sum_{m=-\infty}^{\infty} \left| G\left(\frac{m}{T}\right)\right|^2 \delta\left(f - \frac{m}{T}\right)$$