Homework-2 solutions.

8.1. $\psi_n(t)$, $n = 1, 2, 3$ are orthogonal. $\Rightarrow$ we have to prove.

$$\int_{-\infty}^{\infty} \psi_m(t)\,\psi_n(t)\,dt = 0 \quad m \neq n$$

(a) $$C_{12} = \int_{-\infty}^{\infty} \psi_1(t)\,\psi_2(t)\,dt = \int_0^4 \psi_1(t)\,\psi_2(t)\,dt$$

$$= \int_0^2 \psi_1(t)\psi_2(t)\,dt + \int_2^4 \psi_1(t)\,\psi_2(t)\,dt$$

$$= \frac{1}{4}\times 2 - \frac{1}{4}\times(4-2) = \underline{0}$$

$$C_{13} = \int_{-\infty}^{\infty} \psi_1(t)\,\psi_3(t)\,dt = \int_0^4 \psi_1(t)\,\psi_3(t)\,dt$$

$$= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt - \frac{1}{4}\int_2^3 dt + \frac{1}{4}\int_3^4 dt$$

$$= \underline{0}.$$

$$C_{23} = \int_{-\infty}^{\infty} \psi_2(t)\,\psi_3(t)\,dt = \int_0^4 \psi_2(t)\,\psi_3(t)\,dt$$

$$= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt + \frac{1}{4}\int_2^3 dt - \int_3^4 dt$$

$$= \underline{0}.$$

$\therefore$ Signals $\psi_n(t)$ are orthogonal.

2) We find the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t)\,\varphi_n(t)\,dt \quad , \; n = 1, 2, 3.$$

$$x_1 = \int_0^4 x(t)\,\varphi_1(t)\,dt = \frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^2 dt - \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt$$

$$= \underline{\underline{0}}$$

$$x_2 = \int_0^4 x(t)\,\varphi_2(t)\,dt = \frac{1}{2}\int_0^4 x(t)\,dt = 0.$$

$$x_3 = \int_0^4 x(t)\,\varphi_3(t)\,dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt$$

$$= 0.$$

$\therefore$ $x(t)$ is orthogonal to $\varphi_i(t)$, $i = 1, 2, 3$, and thus it cannot be represented as a linear combination of these functions.

$\underline{8.2}$ The expansion coefficients $\{c_n\}$ that minimize the mean square error satisfy

$$c_n = \int_{-\infty}^{\infty} x(t)\,\varphi_n(t)\,dt = \int_0^4 \sin\frac{\pi t}{4}\,\varphi_n(t)\,dt$$

$$c_1 = \int_0^4 \sin\frac{\pi t}{4}\,\varphi_1(t)\,dt = \frac{1}{2}\int_0^2 \sin\frac{\pi t}{4}\,dt - \frac{1}{2}\int_2^4 \sin\frac{\pi t}{4}\,dt$$

$$= -\frac{2}{\pi}\cos\left(\frac{\pi t}{4}\right)\Big|_0^2 + \frac{2}{\pi}\cos\frac{\pi t}{4}\Big|_2^4$$

$$= -\frac{2}{\pi}(0-1) + \frac{2}{\pi}(-1-0) = \underline{\underline{0}}$$

$$c_2 = \int_0^4 \sin\frac{\pi t}{4}\,\varphi_2(t)\,dt = \frac{1}{2}\int_0^4 \sin\frac{\pi t}{4}\,dt$$

$$= -\frac{2}{\pi}\cos\frac{\pi t}{4}\Big|_0^4 = -\frac{2}{\pi}(-1-1) = \underline{\underline{4/\pi}}$$

$$c_3 = \int_0^4 \sin\frac{\pi t}{4}\,\varphi_3(t)\,dt$$

$$= \frac{1}{2}\int_0^1 \sin\frac{\pi t}{4}\,dt - \frac{1}{2}\int_1^2 \sin\frac{\pi t}{4}\,dt + \frac{1}{2}\int_2^3 \sin\frac{\pi t}{4}\,dt - \frac{1}{2}\int_3^4 \sin\frac{\pi t}{4}\,dt$$

$$= \underline{\underline{0}}.$$

2) The residual mean square error E_{min} can be found from

$$E_{min} = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{3} |C_i|^2$$

$$= \int_0^4 \left(\sin \frac{\pi t}{4}\right)^2 dt - \left(\frac{4}{\pi}\right)^2 = \frac{1}{2}\int_0^4 \left(1 - \cos \frac{\pi t}{2}\right) dt - \frac{16}{\pi^2}$$

$$= 2 - \frac{1}{\pi} \sin \frac{\pi t}{2} \Big|_0^4 - \frac{16}{\pi^2} = \underline{\underline{2 - \frac{16}{\pi^2}}}$$

8.5.

1) Impulse response of matched filter to $s(t)$ is

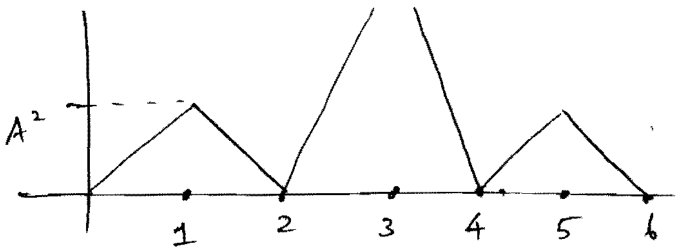
$h(t) = s(T-t) = s(3-t) = s(t)$..

because $s(t)$ is even w.r.t $t = T/2 = 3/2$ axis.

2) The output of the matched filter is

$$y(t) = s(t) * s(t) = \int_0^t s(\tau) s(t-\tau) d\tau$$

$$= \begin{cases} 0, & t < 0 \\ A^2 t, & 0 \le t < 1 \\ A^2(2-t), & 1 \le t < 2 \\ 2A^2(t-2) & 2 \le t < 3. \\ 2A^2(4-t) & 3 \le t < 4 \\ A^2(t-4) & 4 \le t < 5 \\ A^2(6-t) & 5 \le t < 6 \\ 0 & 6 \le t. \end{cases}$$



3) At the output of the matched filter for $t = T = 3$. the noise is

$$n_T = \int_0^T n(t)\, h(T-t)\, dt$$

$$= \int_0^T n(\tau)\, h(T - (T-\tau))\, d\tau = \int_0^T n(\tau) s(\tau)\, d\tau$$

The variance of the noise is

$$\sigma^2_{n_T} = E\left[\int_0^T \int_0^T n(\tau) n(\nu) s(\tau) s(\nu)\, d\tau\, d\nu \right]$$

$$= \int_0^T \int_0^T s(\tau) s(\nu) E[n(\tau) n(\nu)]\, d\tau\, d\nu$$

$$= \frac{N_o}{2} \int_0^T s^2(\tau)\, d\tau = \underline{N_o A^2}$$

4) For antipodal equiprobable signals prob. of error is

$$P(e) = Q\left(\sqrt{\frac{2E_b}{N_o}}\right) \qquad E_b = \underline{2A^2}$$

$$P(e) = \underline{Q\left(\frac{2A}{\sqrt{N_o}}\right)}$$

8.17

The energy of the two signals $s_1(t)$ and $s_2(t)$ is

$$\mathcal{E}_b = A^2 T$$

Basis function

$$\psi(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T/2 \\ -\frac{1}{\sqrt{T}} & T/2 \le t \le T \end{cases}$$

The vector representation of signals is

$$\pm A\sqrt{T}$$

$$P(e) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad \Rightarrow \text{equiprobable antipodal signals.}$$

$$= Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$