Steady state tracking

Recall that integral control gives zero steady state error to a step *even in the presence of plant modelling mismatch.*

This does not happen in our state feedback reference tracking scheme.

$$u(k) \;=\; K(x_{ref} - x(k)),$$

so if $x(k) \longrightarrow x_{ref}$ then $u(k) \longrightarrow 0$.

However with $u(k) \longrightarrow 0$, and no plant poles at $z = 1$, we have, $y(k) \longrightarrow 0$. Clearly then, $y(k) \neq r(k)$, the reference.

Feedforward compensation

Recall that the matrix N_u can provide some steady-state feedforward compensation:

$$u(k) \;=\; K(x_{ref} - x(k)) \;+\; N_u\, r(k),$$

where,

$$N_u \;=\; \frac{1}{\text{plant steady state gain}}.$$

This is only correct if we know the plant steady state gain.

Integral control

Recall the idea behind integral control. If we consider the integral of the error at time, t,

$$\int_0^t (r(\tau) - y(\tau))\, d\tau,$$

we want to make this quantity go zero. In other words,

$$\lim_{t \longrightarrow \infty} \int_0^t (r(\tau) - y(\tau))\, d\tau \;=\; 0.$$

This means that as $t \longrightarrow \infty$, the error must go to zero,

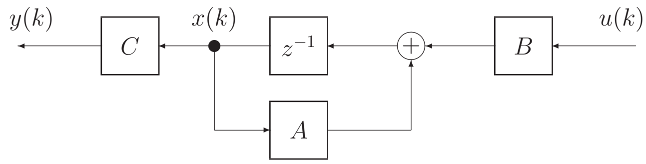
$$y(t) \longrightarrow r(t).$$

If this wasn't the case (i.e. in steady state $y \neq r$), then the integral would end up going to ∞.

Approach

Make the integral of the error $(r(k) - y(k))$ go to zero for state feedback.

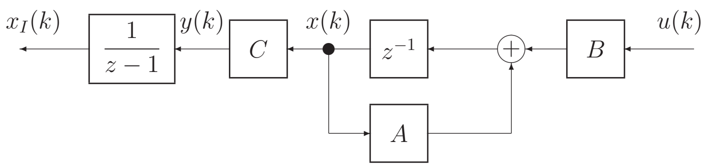
Integral control



State space:

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned}$$

Augment the output, $y(k)$ with an integrator.



New variable: $x_I(k)$

$$\begin{aligned} x_I(k+1) &= x_I(k) + y(k) \\ &= x_I(k) + C\,x(k) \end{aligned}$$

Augmented system

$$\begin{bmatrix} x_I(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} I & C \\ 0 & A \end{bmatrix} \begin{bmatrix} x_I(k) \\ x(k) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(k).$$

This now has $n+1$ states (or, in general, n plus the number of outputs).

Applying state feedback

If we now design a state feedback controller (using pole placement),

$$u(k) = -\begin{bmatrix} K_I & K \end{bmatrix} \begin{bmatrix} x_I(k) \\ x(k) \end{bmatrix},$$

then this will make $x(k) \longrightarrow 0$ and the integrator output go to zero.

$$x_I(k) = \frac{1}{z-1}\,y(k) \longrightarrow 0.$$

Reference tracking

Recall that we replaced $u(k) = -K\,x(k)$ by $u(k) = -K(x_{ref} - x(k))$.

Key idea

Replace $x(k)$ by $x(k) - x_{ref}$ (state error)

and

$\frac{1}{z-1}\,y(k)$ by $\frac{1}{z-1}\,(y(k) - r(k))$ (integral of the tracking error).

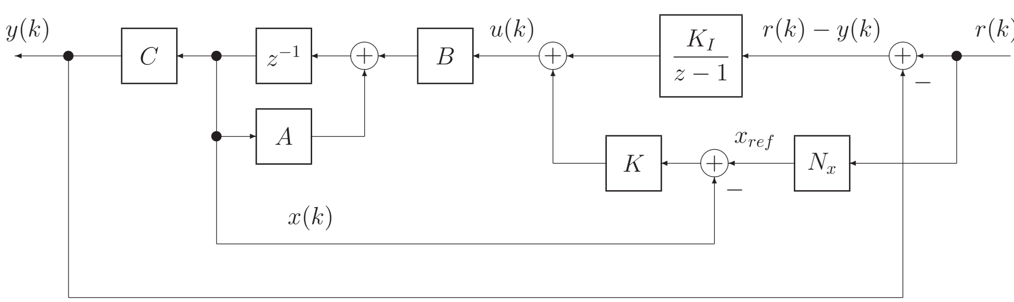
This will make both the state error and the integral of the tracking error to go to zero.

Implementation:

$$u(k) \;=\; -\begin{bmatrix} K_I & K \end{bmatrix} \begin{bmatrix} \frac{1}{z-1}\,(y(k) - r(k)) \\ x(k) - x_{ref} \end{bmatrix} \;=\; \begin{bmatrix} K_I & K \end{bmatrix} \begin{bmatrix} \frac{1}{z-1}\,(r(k) - y(k)) \\ x_{ref} - x(k) \end{bmatrix}.$$

Implementation

$$u(k) \;=\; \begin{bmatrix} K_I & K \end{bmatrix} \begin{bmatrix} \frac{1}{z-1}\,(r(k) - y(k)) \\ x_{ref} - x(k) \end{bmatrix}.$$

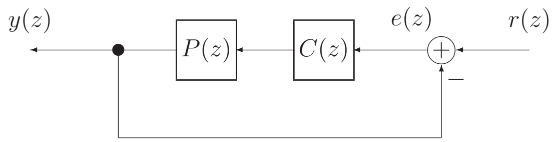


If $x(k)$ is not measured: build an estimator and use $\hat{x}(k)$ in the above.

A feedforward term (N_u) is no longer necessary.

Reference error format

In some cases the controller has access only to the signal, $e(k) = r(k) - y(k)$.



Example: Room thermostats for temperature control.

Consequences?

State feedback/estimator design methods use separate measurements of $y(k)$ and $r(k)$.

Can we still do a state feedback design if we measure only $e(k)$?

Estimation with only $e(k)$

Given a plant,

$$\begin{aligned} x(k+1) &= A\,x(k) \; + \; B\,u(k) \\ y(k) &= C\,x(k), \end{aligned}$$

we build an estimator and state feedback controller via,

$$\begin{aligned} \hat{x}(k+1) &= (A - BK)\,\hat{x}(k) \; + \; L\,(y(k) - C\hat{x}(k)) \\ u(k) &= -K\,x(k), \end{aligned}$$

or, equivalently,

$$\begin{aligned} \hat{x}(k+1) &= (A - BK - LC)\,\hat{x}(k) \; + \; L\,y(k) \\ u(k) &= -K\,x(k). \end{aligned}$$

This isn't quite in the correct form. We cannot use N_x or N_u as $r(k)$ isn't available.

Estimation with only $e(k)$

$$\begin{aligned}\hat{x}(k+1) &= (A - BK - LC)\,\hat{x}(k) \;+\; L\,y(k) \\ u(k) &= -K\,x(k).\end{aligned}$$

If we add an extra term, $-L\,r(k)$, we get,

$$\begin{aligned}\hat{x}(k+1) &= (A - BK - LC)\,\hat{x}(k) \;+\; L\,y(k) - \underbrace{L\,r(k)}_{\text{extra term}} \\ u(k) &= -K\,x(k)\end{aligned}$$

This is now in a form we can implement with only an $e(k)$ measurement.

$$\begin{aligned}\hat{x}(k+1) &= (A - BK - LC)\,\hat{x}(k) \;-\; L\,e(k) \\ u(k) &= -K\,x(k).\end{aligned}$$

The term $-L\,r(k)$ is an unwanted input to the estimator and will cause an offset in the estimation. If $r(k) = 0$ this isn't a problem.