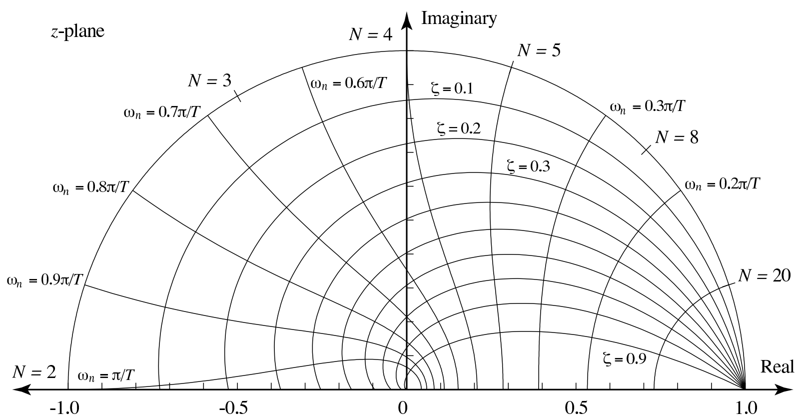
Closed-loop pole locations

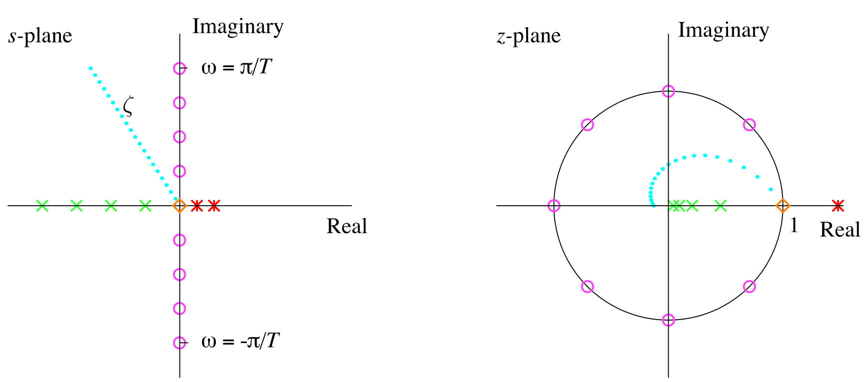
Continuous-time intuition: rise-time, damping ratio, settling time.

Linear Quadratic Regulators (LQR)

Optimal estimation (Kalman filtering)



Closed-loop pole locations

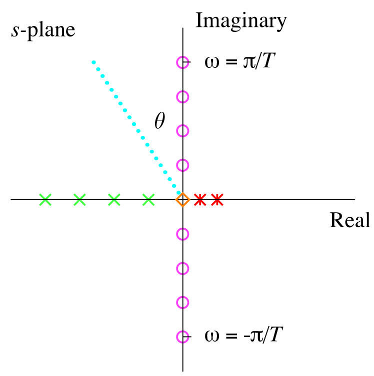


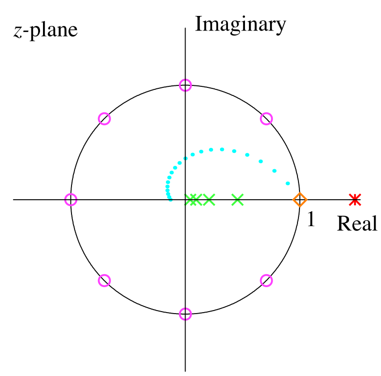
$$x(s) = \frac{1}{1 - s/\alpha} \qquad (\text{pole } = \alpha, \quad \text{time constant} = 1/\alpha = \tau)$$

Settling time (to within 1%): $t_s = \dfrac{4.6}{\alpha} = 4.6\,\tau$

Discrete equivalent: $z_\alpha = e^{\alpha T}$

Closed-loop pole locations





Dominant complex poles: $\hat{s}, \hat{s}^*$

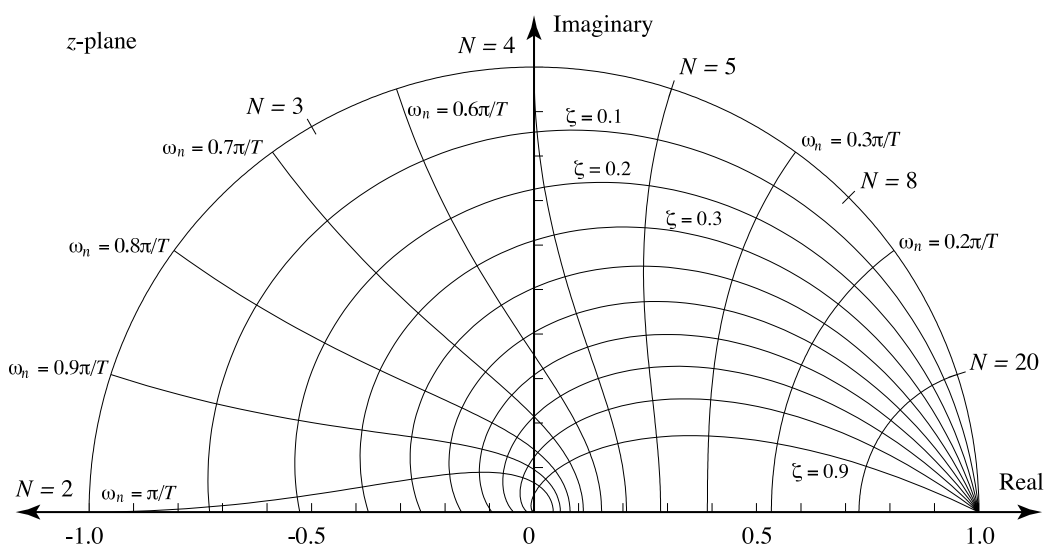
Natural frequency: $\omega_n = \text{abs}(\hat{s})$

Damping ratio: $\zeta = \sin(\theta)$

Rise time (10% to 90%): $t_r \approx \dfrac{1.8}{\omega_n}$

Overshoot: $M = \mathrm{e}^{-\pi\zeta/\sqrt{1-\zeta^2}}$

Closed-loop pole locations



Pole placement

State feedback: $u(k) = -K\,x(k)$

Closed-loop dynamics: $x(k+1) = (A - BK)\,x(k)$

Closed-loop poles given by: $\mathrm{eig}(A - BK)$

Ackermann's formula: $K = [0\ \cdots\ 0\ 1]\,\mathcal{C}^{-1}\gamma_c(A)$

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%  Example by hand ...
%  desired closed-loop poles are p1 and p2

I2 = eye(2,2);
Ctrl = ctrb(A,B);                          % controllability matrix
gamma_c = (A - p1*I2) * (A - p2*I2);      % polynomial in A
K = [0 , 1] * inv(Ctrl) * gamma_c;


%  Example with place

K = place(A,B, [p1; p2]);
```


Linear Quadratic Regulator (LQR)

State feedback: $u(k) = -K\,x(k)$

Closed-loop dynamics: $x_{\text{clp}}(k+1) = (A - BK)\,x_{\text{clp}}(k)$

Closed-loop poles given by: $\mathrm{eig}(A - BK)$

Define a cost function: $J_{\text{LQR}} = \sum_{k=1}^{\infty} x_{\text{clp}}(k)^T\,Q\,x_{\text{clp}}(k) + u(k)^T\,R\,u(k)$

Weighting matrices: Q (state) and R (input)

Find the K that minimizes J_{LQR}

Note that this guarantees that $A - BK$ is stable. Why?

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K = dlqr(A,B,Q,R);
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Linear Quadratic Regulator (LQR)

$$J_{\rm LQR} \;=\; \sum_{k=1}^{\infty} x_{\rm clp}(k)^T\, Q\, x_{\rm clp}(k) \;+\; u(k)^T\, R\, u(k)$$

Weighting matrices: Q (state) and R (input)

Q is symmetric and positive definite: $Q = Q^T$ and $x^T Q x > 0$ for all x

R is symmetric and positive definite: $R = R^T$ and $u^T R u > 0$ for all u

$Q \in \mathcal{R}^{\dim(x) \times \dim(x)}$ and $R \in \mathcal{R}^{\dim(u) \times \dim(u)}$

Initial choice (Bryson's rule): Q is diagonal with $Q_{ii} \;=\; \dfrac{1}{\max\{x_i(k)^2\}}$

R is diagonal with $R_{ii} \;=\; \dfrac{1}{\max\{u_i(k)^2\}}$

Linear Quadratic Estimation (Kalman filtering)

How to choose pole locations for the estimation errors: $A \;-\; LC$

Plant model:

$$\begin{aligned} x(k+1) &= A\,x(k) \;+\; B\,u(k) \;+\; G\,w(k) \\ y(k) &= C\,x(k) \;+\; v(k) \end{aligned}$$

Process noise: $w(k)$ (zero-mean white noise)

Measurement noise: $v(k)$ (zero-mean white noise)

Estimator: $\hat{x}(k+1) \;=\; A\,\hat{x}(k) \;+\; B\,u(k) \;+\; L(y(k) \;-\; C\hat{x}(k))$

Estimator error: $\tilde{x}(k) \;=\; x(k) \;-\; \hat{x}(k)$

$$\tilde{x}(k+1) \;=\; (A \;-\; LC)\,\tilde{x}(k) \;+\; G\,w(k) \;-\; L\,v(k)$$

Linear Quadratic Estimation (Kalman filtering)

Plant model:
$$\begin{aligned} x(k+1) &= A\,x(k) \;+\; B\,u(k) \;+\; G\,w(k) \\ y(k) &= C\,x(k) \;+\; v(k) \end{aligned}$$

What is the optimal choice of L to balance between the two noise sources: $w(k)$ and $v(k)$?

Estimation error: $\tilde{x}(k+1) \;=\; (A \;-\; LC)\,\tilde{x}(k) \;+\; G\,w(k) \;-\; L\,v(k)$

Objective: $\min \quad \mathcal{E}\{\tilde{x}(k)^T \tilde{x}(k)\}$ (minimize estimation error variance)

To do this we must know (or at least approximate):

$$\mathcal{E}\left\{w(k)\,w(k)^T\right\} \;=\; Q_N \qquad \left(Q_N \;\in\; \mathcal{R}^{\dim(w)\,\times\,\dim(w)}\right)$$

$$\mathcal{E}\left\{v(k)\,v(k)^T\right\} \;=\; R_N \qquad \left(R_N \;\in\; \mathcal{R}^{\dim(y)\,\times\,\dim(y)}\right)$$

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L = kalman(Pz,QN,RN)
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