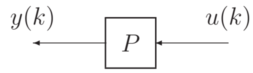
Discrete-time Systems:



Input and output signals are sequences:

$$u = \{u(0), u(1), u(2), \ldots, u(k), \ldots\} \qquad \text{and} \qquad y = \{y(0), y(1), y(2), \ldots, y(k), \ldots\}.$$

Causal LTI/LSI models can be described by difference equations:

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) \ \ldots - a_n y(k-n) \ + b_0 u(k) + b_1 u(k-1) \ \ldots \ b_m u(k-m).$$

Note that the current output, $y(k)$, depends only on current and past inputs, $u(k)$, $u(k-1)$, $\ldots$, and past outputs, $y(k-1)$, $\ldots$.

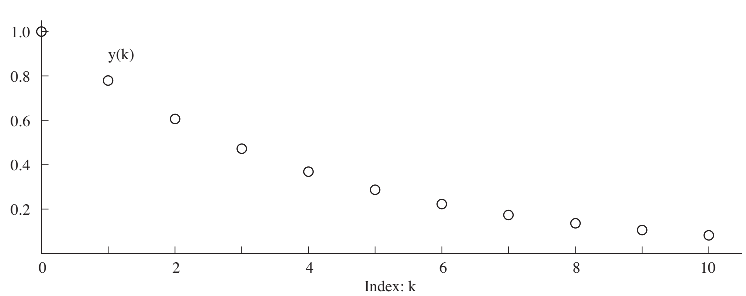
Z-Transform

Recall the Z-Transform definition:

$$y(z) := \sum_{k=-\infty}^{\infty} y(k) z^{-k},$$

where $z \in \mathcal{C}$. This has an associated region of convergence: $r_0 < |z| < R_0$.

Example: $y(k) = \begin{cases} 0 & \text{for } k < 0, \\ e^{-akT} & \text{for } k \geq 0 \end{cases}$



$$\begin{aligned} a &= 0.25, \\ T &= 1 \end{aligned}$$

Example (continued)

$$
\begin{aligned}
y(z) &= \sum_{k=0}^{\infty} \mathrm{e}^{-akT} z^{-k} \quad = \quad \sum_{k=0}^{\infty} \left(\mathrm{e}^{-aT} z^{-1}\right)^k \\
&= \frac{1}{1-\mathrm{e}^{-aT} z^{-1}}, \qquad \text{for } \mathrm{e}^{-aT} < |z| < \infty, \\
&= \frac{z}{z-\mathrm{e}^{-aT}}, \qquad \text{for } \mathrm{e}^{-aT} < |z| < \infty.
\end{aligned}
$$

The region of convergence is needed to reconstruct the signal. (Exercise: reconstruct this Z-tranform for $|z| < \mathrm{e}^{-aT}$)

Shifted sequences

$$
\begin{aligned}
\mathcal{Z}\{y(k-l)\} &= \sum_{k=-\infty}^{\infty} y(k-l) z^{-k}, \\
&= \sum_{i=-\infty}^{\infty} y(i) z^{-i} z^{-l}, \text{by substituting } i = k-l, \\
&= z^{-l} \sum_{i=\infty}^{\infty} y(i) z^{-i} \\
&= z^{-l} y(z).
\end{aligned}
$$

Transfer Functions

Applying Z-transforms to the difference equations gives:

$$
\begin{aligned}
y(z) &= \sum_{k=-\infty}^{\infty} y(k) z^{-k} \\
&= \sum_{k=-\infty}^{\infty} \left(-a_1 y(k-1) \ldots - a_n y(k-n) + b_0 u(k) \ldots + b_m u(k-m)\right) z^{-k}, \\
&= -a_1 \sum_{k=-\infty}^{\infty} y(k-1) z^{-k} \ldots - a_n \sum_{k=-\infty}^{\infty} y(k-n) z^{-k} + b_0 \sum_{k=-\infty}^{\infty} u(k) z^{-k} \ldots + b_m \sum_{k=-\infty}^{\infty} u(k-m) z^{-k}, \\
&= -a_1 z^{-1} y(z) \ldots - a_n z^{-n} y(z) + b_0 u(z) \ldots b_m z^{-m} u(z),
\end{aligned}
$$

Rearranging gives,

$$
\left(1 + a_1 z^{-1} + \cdots + a_n z^{-n}\right) y(z) = \left(b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}\right) u(z).
$$

From this we get the transfer function,

$$
P(z) = \frac{y(z)}{u(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \cdots + a_n} = \frac{b(z)}{a(z)}.
$$

Pole/Zero locations

$$P(z) = \frac{b(z)}{a(z)}$$

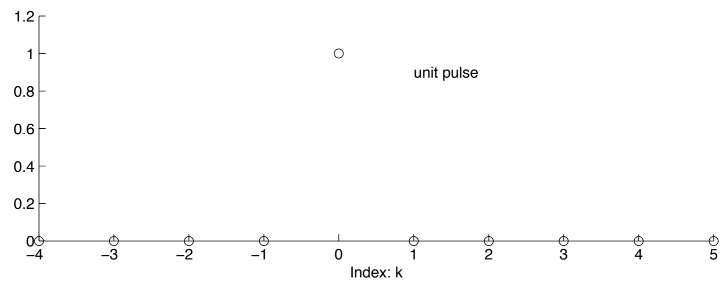
The pole and zero positions determine the typical plant responses.

The poles are the roots of $a(z) = 0$.

The zeros are the roots of $b(z) = 0$.

Unit pulse response: $u(k) = \begin{cases} 0 & \text{for } k < 0, \\ 1 & \text{for } k = 0, \\ 0 & \text{for } k > 0 \end{cases}$

This is used to characterise LTI discrete-time systems (cf. impulse response)



Unit pulse response

Note that if $u(k)$ is the unit pulse then,

$$u(z) = \sum_{k=-\infty}^{\infty} u(k) z^k = 1z^0 = 1.$$

So the system pulse response is given by,

$$y(k) = \mathcal{Z}^{-1}\{y(z)\} = \mathcal{Z}^{-1}\{P(z)u(z)\} = \mathcal{Z}^{-1}\{P(z)\},$$

which is the inverse Z-transform of the transfer function. (cf. continuous-time).

First order example:

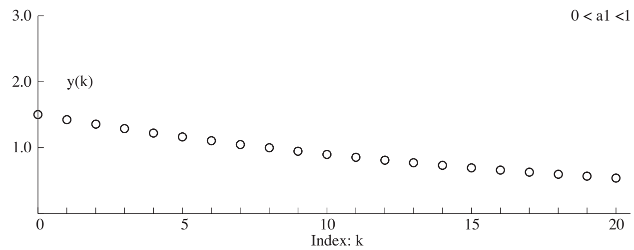
$$P(z) = \frac{b_1 z}{z - a_1}, \qquad \text{and so the pulse response is} \qquad y(k) = \mathcal{Z}^{-1}\{P(z)\} = b_1\, a_1^k.$$

We will use this example to look at the various behaviors possible.

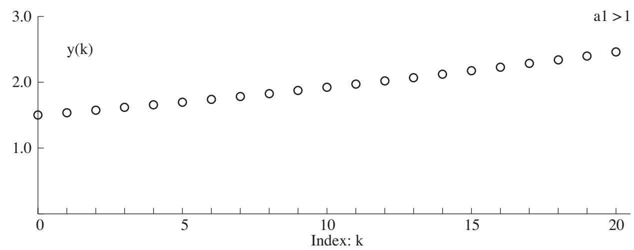
First order example continued

The examples are illustrated for $b_1 = 1.5$.

$0 < a_1 < 1$
(inside the unit circle)

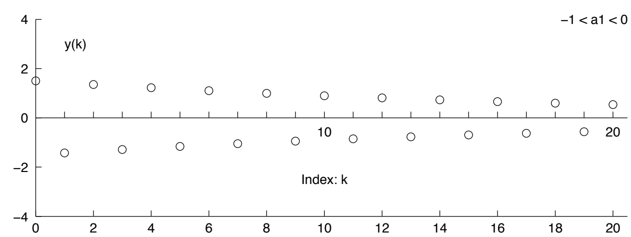


$a_1 > 1$
(outside the unit circle)

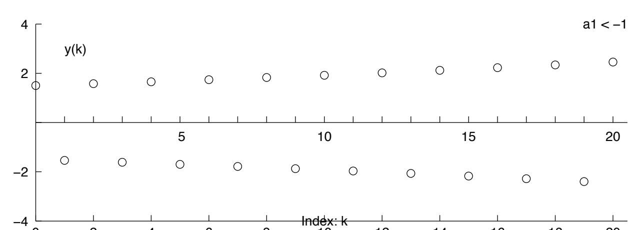


First order example continued: $(a_1 < 0)$

$-1 < a_1 < 0$
(inside the unit circle)



$a_1 < -1$
(outside the unit circle)



We can see that for $|a_1| < 1$ the responses decay. If $|a_1| > 1$ the responses blow up. If a_1 is negative the responses alternate in sign.

Generic second order system

$$P(z) = \frac{N(z^{-1})}{1 + a_1 z^{-1} + a_2 z^{-2}},$$

where $N(z^{-1})$ is a numerator polynomial in terms of z^{-1}.

By partial fraction expansion,

$$P(z) \quad = \quad \frac{B}{1 - pz^{-1}} + \frac{B^*}{1 - p^* z^{-1}} \quad = \quad \frac{(\alpha + j\beta)}{1 - r\mathrm{e}^{j\theta} z^{-1}} + \frac{(\alpha - j\beta)}{1 - r\mathrm{e}^{-j\theta} z^{-1}}$$

This assumes the poles are complex conjugates: $r\mathrm{e}^{j\theta}$ and $r\mathrm{e}^{-j\theta}$.

The unit pulse response is,

$$\begin{aligned} y(k) = \mathcal{Z}^{-1}\{P(z)\,u(z)\} &= \mathcal{Z}^{-1}\left\{ \left(\frac{B}{1 - pz^{-1}} + \frac{B^*}{1 - p^* z^{-1}} \right) 1 \right\} \\ &= (\alpha + j\beta) r^k \mathrm{e}^{jk\theta} + (\alpha - j\beta) r^k \mathrm{e}^{-jk\theta} \\ &= r^k \left[\alpha \left(\mathrm{e}^{jk\theta} + \mathrm{e}^{-jk\theta} \right) + j\beta \left(\mathrm{e}^{jk\theta} - \mathrm{e}^{-jk\theta} \right) \right] \\ &= r^k (2\alpha \cos k\theta - 2\beta \sin k\theta). \end{aligned}$$

Growth/decay rate: depends on r. Frequency of oscillation: depends on θ.

Generic second order system

For example, consider the pulse response:

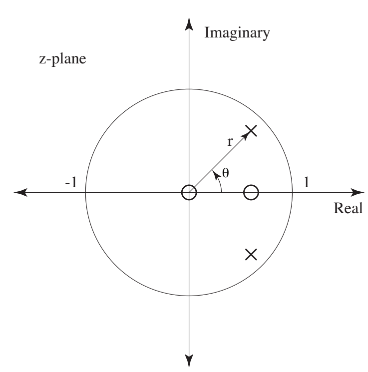
$$y(k) = r^k \cos k\theta, \quad \text{for } k \geq 0, \quad \text{with } r > 0.$$

Which corresponds to

$$P(z) \quad = \quad \frac{1 - r\cos\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}} \quad = \quad \frac{z(z - r\cos\theta)}{z^2 - 2r\cos\theta z + r^2}.$$

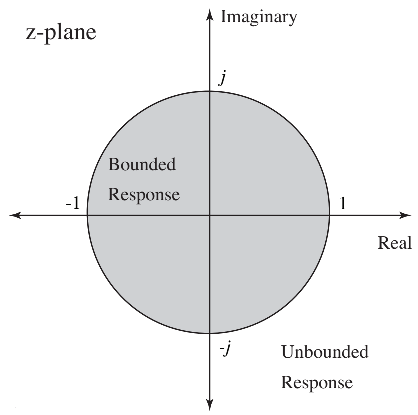
Pole/zero pattern

- $r < 1$: response decays
- θ: determines #samples/oscillation
- $\theta = \pi/4$ (shown) implies 8 samples per oscillation



Decay rates

If $r < 1$ then the response is bounded and decays.



Number of samples to decay to 1% of initial value:

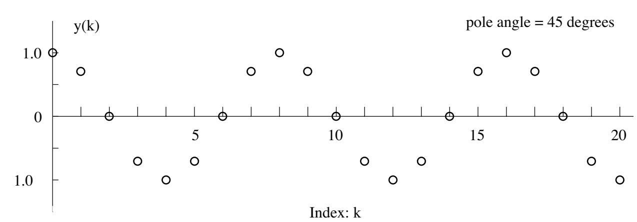
Pole radius r	Decay duration (# samples)
0.9	43
0.8	21
0.6	9
0.4	5

Oscillatory responses: Pole angle (θ) determines how oscillatory the response will be.

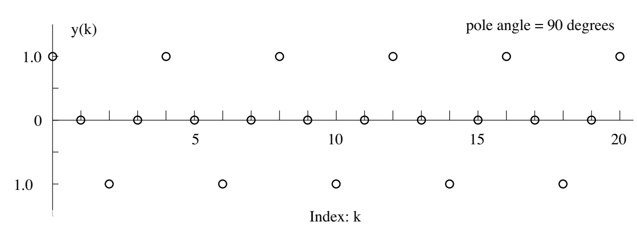
Periodic responses (with period N): $\cos(\theta k) = \cos(\theta(k+N))$ or $N = \dfrac{2\pi}{\theta}$.

Examples

$\theta = \pi/4$ implies
$N = 8$ samples/oscillation



$\theta = \pi/2$ implies
$N = 4$ samples/oscillation

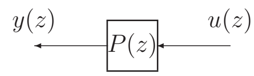


Discrete-time System Stability

$P(z)$ has pulse response $p(k)$,

$$p(k) = \mathcal{Z}^{-1}\{P(z)\}.$$

The output, $y(k)$, is given by the discrete-time convolution,



$$y(k) = \sum_{l=-\infty}^{\infty} p(k-l)u(l).$$

BIBO stability

If $|u(k)| < \infty$, under what conditions is $|y(k)| < \infty$?

BIBO Stability (sufficiency)

Assume that $u(l)$ is bounded: $|u(l)| \leq M < \infty$, for all l.

Now look at,

$$|y(k)| = \left| \sum_{l=-\infty}^{\infty} p(k-l)u(l) \right| \leq \sum_{l=-\infty}^{\infty} |p(k-l)|\,|u(l)|.$$

As $|u(l)| \leq M$,

$$|y(k)| \leq M \sum_{l=-\infty}^{\infty} |p(k-l)|.$$

So if

$$\sum_{l=-\infty}^{\infty} |p(l)| < \infty,$$

then the system is BIBO stable.

BIBO Stability (necessity)

If $\displaystyle\sum_{l=-\infty}^{\infty} |p(l)| = \infty,$ is the system necessarily unstable?

Does there exist a particular $u(k)$ which makes $|y(l)| = \infty$ for some particular l?

Consider: $$u(-l) = \begin{cases} \dfrac{p(l)}{|p(l)|} & \text{if } p(l) \neq 0, \\ 0 & \text{if } p(l) = 0 \end{cases}.$$

This is a sequence made up of 0, 1, and -1. It is bounded ($M = 1$).

Now look at $y(0)$.

$$y(0) = \sum_{l=-\infty}^{\infty} p(-l)u(l) = \sum_{l=-\infty}^{\infty} \frac{(p(-l))^2}{|p(-l)|} = \sum_{l=-\infty}^{\infty} |p(-l)| = \sum_{l=-\infty}^{\infty} |p(l)| = \infty.$$

Unbounded!