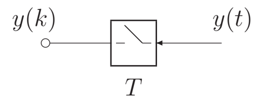
Sampling: period $= T$



Example (single pole signal)

Consider, $\quad y(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0 & t < 0 \end{cases} \quad$ with $\quad a > 0.$

Laplace transform: $\quad y(s) = \dfrac{1}{s+a}.$

Sampled signal: $\quad y(k) = y(t)\Big|_{t=kT} = e^{-akT} = \left(e^{-aT}\right)^k.$

Z-transform, $\quad y(z) = \dfrac{z}{z - e^{-aT}}.$

The s-plane pole is at $s_1 = -a$, and the corresponding z-plane pole is at $z_1 = e^{-aT}$.

Example: (second order)

Now consider a damped sinusoidal signal, $\quad y(t) = e^{-\alpha t}\sin(\beta t), \quad t \geq 0$, with $\alpha > 0$.

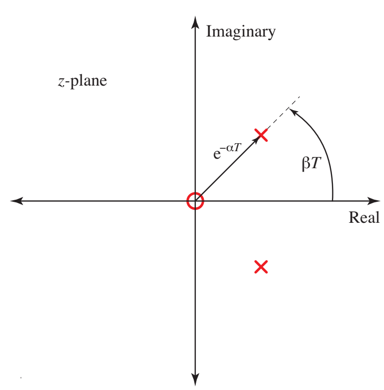
Laplace transform: $\quad y(s) = \dfrac{\beta}{(s+\alpha)^2 + \beta^2}, \quad$ Poles: $s_{1,2} = -\alpha \pm j\beta.$

Sampled signal: $\quad y(k) = e^{-\alpha kT}\sin(\beta kT), \quad k \geq 0.$

Z-transform: $\quad y(z) = \dfrac{z^{-1}e^{-\alpha T}\sin(\beta T)}{1 - z^{-1}2e^{-\alpha T}\cos(\beta T) + z^{-2}e^{-2\alpha T}}.$

Z domain poles given by: $z^2 - 2e^{-\alpha T}\cos(\beta T)z + e^{-2\alpha T} = 0.$

$$\begin{aligned} z_{1,2} &= e^{-\alpha T}\cos(\beta T) \pm \sqrt{e^{2\alpha T}\cos^2(\beta T) - e^{-2\alpha T}} \\ &= e^{-\alpha T}\left(\cos(\beta T) \pm j\sqrt{1 - \cos^2(\beta T)}\right) \\ &= e^{-\alpha T}(\cos(\beta T) \pm j\sin(\beta T)) \\ &= e^{-\alpha T}e^{\pm j\beta T} \\ &= e^{(-\alpha \pm j\beta)T}. \end{aligned}$$

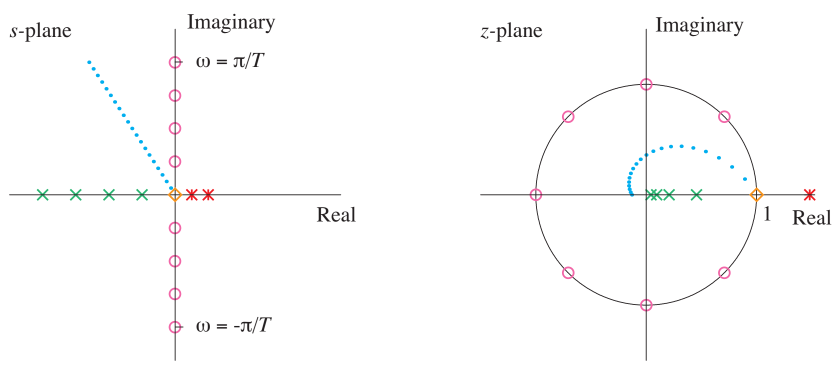


General case:

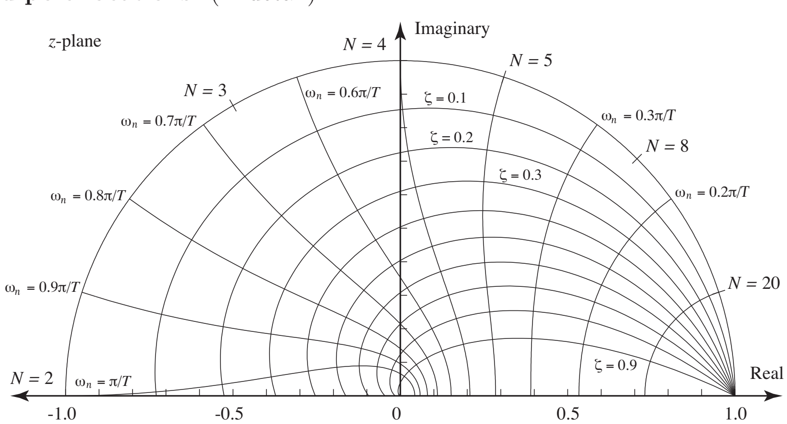
Sampling maps the s-domain poles to the z-domain via: $z_i = e^{s_i T}$.

Stable continuous-time signals ($\text{Re}\{s_i\} < 0$) map to stable discrete-time signals ($|z_i| < 1$).

Pole locations under sampling:



Sampled pole locations: (in detail)

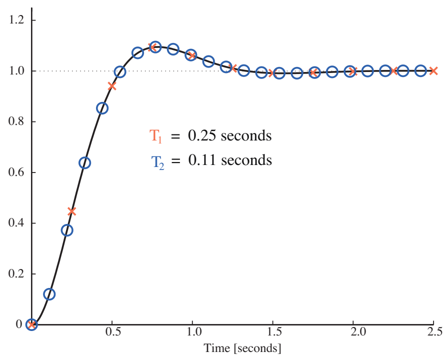


Changing the sampling frequency. (recall that $z_i = e^{s_i T}$.)

Decreasing T: decrease decay rate ($r \rightarrow 1$)
decrease oscillation frequency ($\theta \rightarrow 0$)
poles track constant damping curves towards 1

This is simply because there are more samples taken in the same time period.

Sample rate effects: $z_i = \mathrm{e}^{s_i T}$, changing T changes the pole positions.

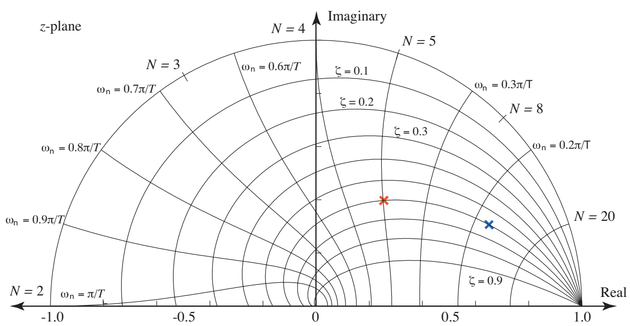


Continuous closed-loop system step response:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\zeta = 0.6$,

$\omega_n = 5$ rad./sec.,
$s_{1,2} = -3 \pm 4i$.



Discrete pole positions:

Period T_1: $z_{1,2} = 0.255 \pm 0.398i$
Period T_2: $z_{1,2} = 0.650 \pm 0.306i$

Aliasing: What happens to signals of high frequencies ($\omega > \pi/T$) ?

As $z_i = \mathrm{e}^{s_i T}$, sinusoids of frequencies from -π/T to π/T radians/second are mapped onto the unit disk by sampling.

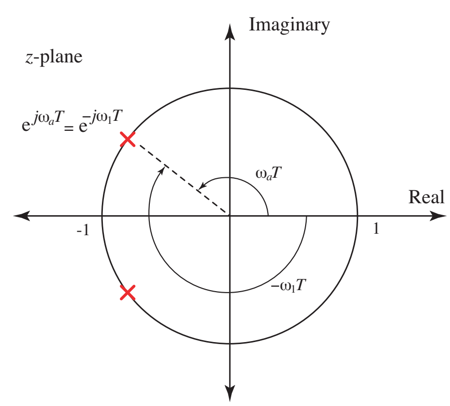
Consider $y(t) = \sin \omega_1 t$, which has Laplace transform: $y(s) = \dfrac{\omega_1}{s^2 + \omega_1^2}$.
Poles are $s_{1,2} = \pm j\omega_1$.

Sample at period T: $y(k) = \sin \omega_1 k T$,

Z-transform: $y(z) = \dfrac{z \sin w_1 T}{z^2 - 2\cos \omega_1 T z + 1}$.

Poles of $y(z)$ are $z_{1,2} = \mathrm{e}^{\pm j\omega_1 T}$.

Slow sampling, $T > \pi/\omega_1$, implies that $\omega_1 T > \pi$.
The pole angle is greater than π.



Having $w_1 T > \pi$, means that,

$$e^{-j\omega_1 T} = e^{j(2\pi - \omega_1 T)}, \quad \text{and} \quad e^{j\omega_1 T} = e^{-j(2\pi - \omega_1 T)}.$$

Now, if $(2\pi - \omega_1 T)$ lies in the range 0 to π radians, the pole pattern is identical to that of a sinusoid of a lower frequency, ω_a, where $\omega_a T = 2\pi - \omega_1 T$.

Equivalently, the apparent frequency is, $\omega_a = \dfrac{2\pi}{T} - \omega_1$. (sampling freq: $2\pi/T$ rad/sec).

Example:

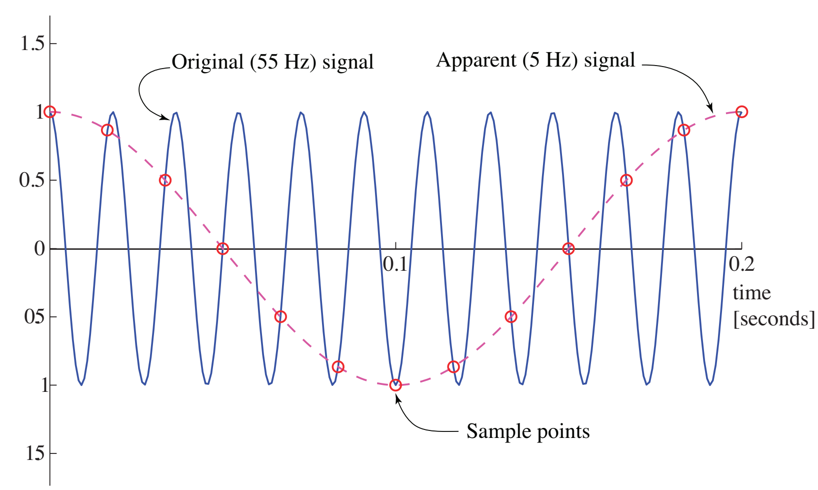
55 Hz Signal: $y(t) = \cos(2\pi 55 t)$

Sampling frequency: $1/T = 60$ Hz

Then $y(k) = \cos(2\pi 55 t)|_{t=kT} = \cos(2\pi 5 t)|_{t=kT}$

Indistinguishable from a sampled 5 Hz signal!

Example: 55 Hz signal sampled at 60 Hz

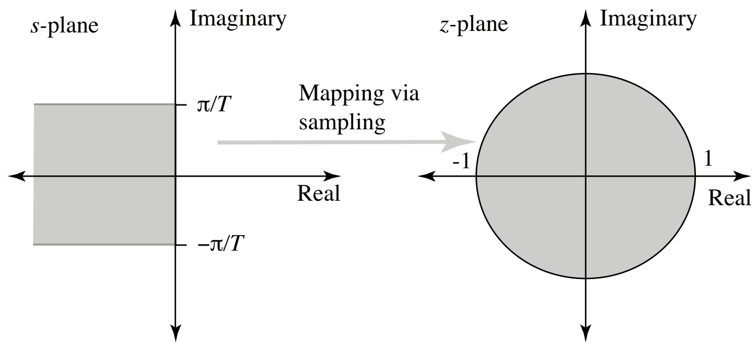


Sampled signals

The unit disk can only represent signals of frequency up to 1/2 the sampling frequency. (**Nyquist frequency**).

Sampling operation maps signal poles via: $z_i = e^{s_i T}$.

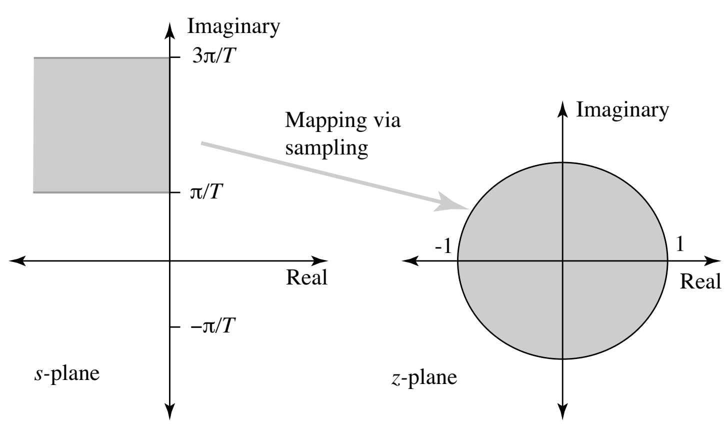
Maps the horizontal strip from $-j\pi/T$ to $j\pi/T$ onto the whole z-plane.



And Re $\{s\} < 0$ in this strip maps to the inside of the unit disk.

Aliasing: (ambiguous mapping of higher frequency signals)

Sampling also maps the next strip (from $j\pi/T$ to $j3\pi/T$) onto the whole z-plane and adds it into the result.

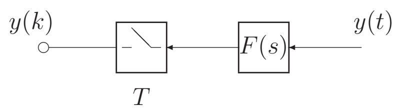


Also true for all (infinite) $2\pi/T$ wide strips above and below the lowest frequency strip.

Consequences of aliasing:

- Ambiguity. Our computer/controller cannot distinguish between frequencies inside the $-\pi/T$ to π/T range and those outside of it.
  - Controller will respond incorrectly to an aliased signal (e.g. disturbance or error).
  - An aliased signal cannot be reconstructed (signal processing).

Amelioration of the problem:



- Anti-aliasing filter. Low pass, rejecting $|\omega| > \pi/T$.
  - High frequency signals no longer enter loop erroneously.
  - High frequency disturbances/errors are "invisible."
  - Filter adds phase lag to the loop. (**Potentially destabilizing!**)