**Approach:**

- Represent the plant, $P(z)$ (or $P(s)$) as an $n$th order differential equation.
- Represent $n$th order differential equation as a 1st order matrix differential equation with dimension $n$.
- Design methods now involve linear algebra.
- Easy to handle large systems (with MATLAB).
- Easy to handle systems with multiple inputs and outputs.
- Easy to simulate systems.

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k), \\
y(k) &= C\,x(k) + D\,u(k).
\end{aligned}
$$

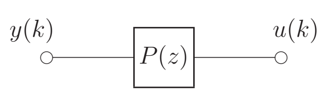
$A$, $B$, $C$ and $D$ can be matrices. $x(k)$ is a vector (state vector).



*State-space systems*

**Representations:** From transfer function to state-space.

Consider a linear, shift invariant, system:




We can express this as a transfer function,

$$
y(z) \quad = \quad P(z)\,u(z) \quad = \quad \frac{b(z)}{a(z)}\,u(z)
$$

where $a(z)$ and $b(z)$ are polynomials, so,

$$
P(z) \quad = \quad \frac{b(z)}{a(z)} \quad = \quad \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}.
$$

For causal systems the order of $b(z)$ is less than or equal to the order of $a(z)$. So $m \leq n$ above.

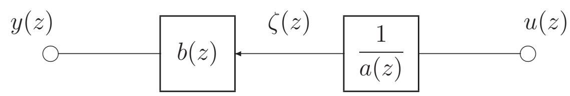
Assume for now that $m < n$,

**Outline:**

1. Draw the system as an interconnected "chain of delays",
2. Relabel the signals in the system,
3. Rewrite the input/output equations in terms of the new signals,
4. Abbreviate the equations to a matrix form (state-space).

Drawing a digital system block diagram in terms of delays is exactly the same as drawing a continuous system block diagram in terms of integrators.

Split the system,




$$\zeta(z) \quad = \quad \frac{1}{a(z)}\, u(z) \qquad \text{and} \qquad y(z) \quad = \quad b(z)\,\zeta(z).$$





**Chain of delays:**

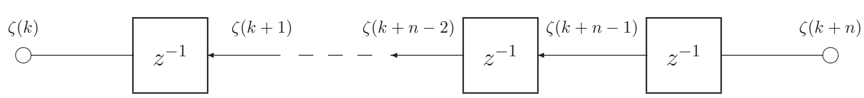
Consider the first equation: $\quad \zeta(z) \quad = \quad \frac{1}{a(z)}\, u(z)$

We want to develop a chain of delay model to get $\zeta(k)$:

**First step:** Write the expression for $n$ delays:

$$\begin{aligned} \zeta(k) &= z^{-1}\,\zeta(k+1) \\ \vdots \quad & \quad \vdots \\ \zeta(k+n-1) &= z^{-1}\zeta(k+n) \end{aligned}$$

In pictures ...




**Second step:** Express $\zeta(k+n)$ in terms of $\zeta(k), \ldots, \zeta(k+n-1)$ and $u(k)$.

**Second step:** Express $\zeta(k+n)$ in terms of $\zeta(k), \ldots, \zeta(k+n-1)$ and $u(k)$.

To do this, write this as: $\zeta(z)\, a(z) = u(z)$,

Expanding $a(z)$ gives,

$$\left(z^n + a_1 z^{n-1} + \cdots + a_n\right) \zeta(z) = u(z)$$

and expressing this in terms of the highest power of $z$ gives:

$$z^n \zeta(z) = u(z) - \left(a_1 z^{n-1} + \cdots + a_n\right) \zeta(z).$$

Now write this is the time domain,

$$\zeta(k+n) = u(k) - \left(a_1 \zeta(k+n-1) + \cdots + a_n \zeta(k)\right).$$

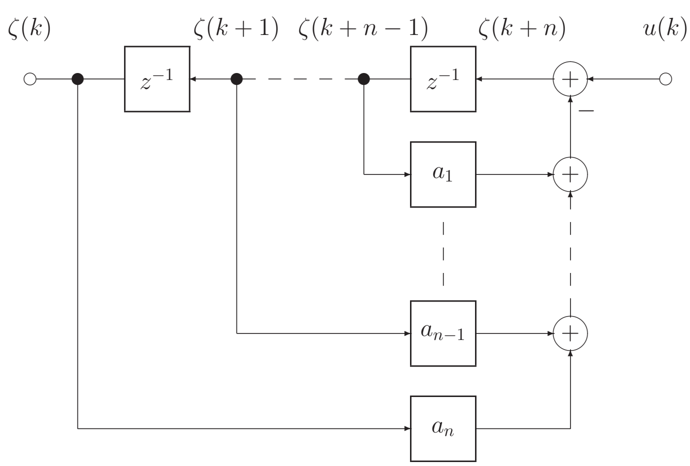
This is the form we need to include in our block diagram.





**Block diagram of:** $\zeta(z) = \dfrac{1}{a(z)} u(z)$

$$\zeta(k+n) = u(k) - \left(a_1 \zeta(k+n-1) + \cdots + a_n \zeta(k)\right).$$



**Numerator term:** $y(z) = b(z)\,\zeta(z)$

Expanding the $b(z)$ polynomial gives,

$$\begin{aligned} y(z) &= b(z)\,\zeta(z), \\ &= \left(b_0 z^m + b_1 z^{m-1} + \cdots + b_m\right)\,\zeta(z) \end{aligned}$$

and in the time domain this is,

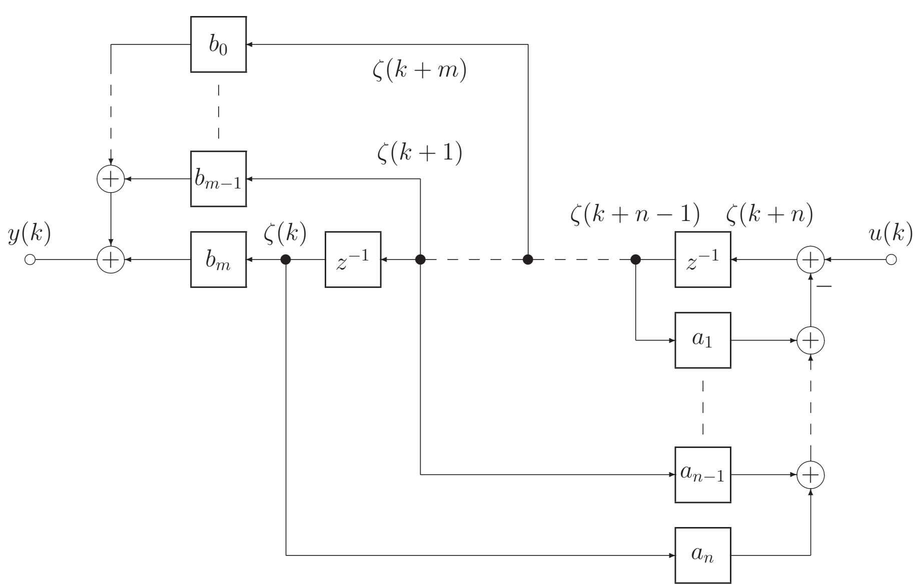
$$y(k) = b_0\zeta(k+m) + b_1\zeta(k+m-1) + \cdots + b_m\zeta(k).$$

As $m < n$ all of the signals $\zeta(k)$ to $\zeta(k+m)$ are available along to the top of the chain of delays.

So $y(k)$ is just a linear combination of signals from the previous block diagram.








This could actually be constructed from shift registers, summers and multipliers.

**What if** $m = n$ **?**

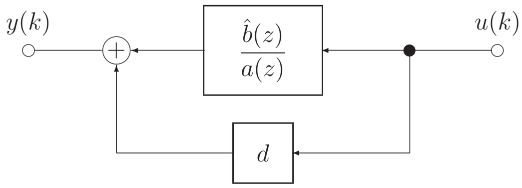
Remove the highest order terms by polynomial division,

$$\frac{b(z)}{a(z)} = d + \frac{\hat{b}(z)}{a(z)}.$$

Now $d$ is a constant and $\hat{b}(z)$ has order $m-1 < n$.

We deal with $\frac{\hat{b}(z)}{a(z)}$ exactly as before and then add the constant, $d$, to the result.

In pictures ...








**Developing the matrix equations**

Relabel our intermediate variables,

$$\begin{aligned} x_1(k) &= \zeta(k+n-1) \\ x_2(k) &= \zeta(k+n-2) \\ &\vdots \\ x_n(k) &= \zeta(k) \end{aligned}$$

Now work out what happens to each of them at time $k+1$:

$$\begin{aligned} x_1(k+1) &= \zeta(k+n) &&= u(k) - (a_1\zeta(k+n-1) + \cdots + a_n\zeta(k)) \\ & &&= u(k) - (a_1 x_1(n) + \cdots + a_n x_n(k)) \\ x_2(k+1) &= \zeta(k+n-2+1) &&= x_1(k) \\ \vdots & \quad \vdots && \quad \vdots \\ x_n(k+1) &= \zeta(k+1) &&= x_{n-1}(k) \end{aligned}$$

Now look at this as a matrix multiplication.

**Matrix equations**

$$
\begin{array}{rcl}
x_1(k+1) & = & -a_1\, x_1(k) \;\; -a_2\, x_2(k) \;\; \cdots \;\; -a_{n-1}\, x_{n-1}(k) \;\; -a_n\, x_n(k) \;\; + \;\; u(k) \\
x_2(k+1) & = & x_1(k) \\
x_3(k+1) & = & x_2(k) \\
\vdots & & \\
x_n(k+1) & = & x_{n-1}(k)
\end{array}
$$

or ...

$$
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} \quad + \quad \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)
$$

Define the state: $x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$ to get the final equations.





**Matrix equations**

$$
x(k+1) \quad = \quad A\, x(k) \quad + \quad B\, u(k),
$$

where

$$
A \quad = \quad \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad B \quad = \quad \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

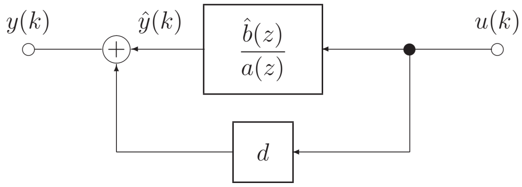
This is the “state-update” equation (or sometimes just the “state”equation).

**What about the output $y(k)$?**

$$
\begin{array}{rcl}
y(k) & = & b_0\zeta(k+m) \;\; +b_1\zeta(k+m-1) \;\; \cdots \;\; +b_m\zeta(k) \\
 & = & b_0 x_{n-m}(k) \;\; +b_1 x_{n-m-1}(k) \;\; \cdots \;\; +b_m x_n(k) \\
 & = & C\, x(k)
\end{array}
$$

where $C = \begin{bmatrix} 0 & \ldots & 0 & b_0 & \ldots & b_m \end{bmatrix}$. If $m = n-1$ then there are no leading zeros in $C$.

**Equal numerator and denominator order case:**




We can calcuate the state-space representation for $\dfrac{\hat{b}(z)}{a(z)}$:

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k), \\ \hat{y}(k) &= C\,x(k), \end{aligned}$$

and as, $y(k) = \hat{y}(k) + d\,u(k)$, we have,

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k), \\ y(k) &= C\,x(k) + d\,u(k). \end{aligned}$$





**Other domains:**

Discrete time, time invariant:

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k), \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

Continuous time, time invariant:

$$\begin{aligned} \frac{d\,x(t)}{dt} &= A\,x(t) + B\,u(t), \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

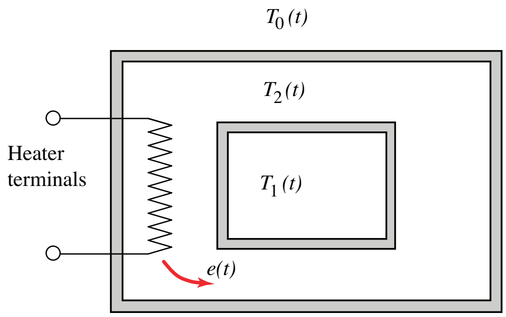
Nonlinear, time invariant:

$$\begin{aligned} \frac{d\,x(t)}{dt} &= f(x(t), u(t)), \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

Nonlinear, time varying:

$$\begin{aligned} \frac{d\,x(t)}{dt} &= f(t, x(t), u(t)), \\ y(t) &= g(t, x(t), u(t)) \end{aligned}$$

**Examples:** A thermal control system:




| | |
|---|---|
| $T_i(t)$ | temperature of volume $i$ |
| $e(t)$ | energy into the volume 1 |
| $m_i$ | mass of volume $i$ |
| $c_i$ | specific heat for volume $i$ |
| $k_{ij}$ | thermal conductance for $i, j$ interface |

We can derive the state-space representation directly from the thermal energy equations.

For volume 1: $$m_1 c_1 \frac{d\,T_1(t)}{dt} = k_{12}(T_2(t) - T_1(t))$$

For volume 2: $$m_2 c_2 \frac{d\,T_2(t)}{dt} = -k_{12}(T_2(t) - T_1(t)) \quad - k_{20}(T_2(t) - T_0(t)) \quad + e(t)$$





**Thermal system:**

Suppose that $T_1(t)$ is the output of interest and $e(t)$ and $T_0(t)$ are both inputs.

Now select state variables which will allow us to put this into the generic state-space matrix equation form.

Try $x(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix}$.

Then, rearranging gives,

$$\frac{d\,T_1(t)}{dt} = \frac{-k_{12}}{m_1 c_1} T_1(t) + \frac{k_{12}}{m_1 c_1} T_2(t)$$

$$\frac{d\,T_2(t)}{dt} = \frac{k_{12}}{m_2 c_2} T_1(t) + \frac{-k_{12} - k_{20}}{m_2 c_2} T_2(t) + \frac{k_{20}}{m_2 c_2} T_0(t) + \frac{1}{m_2 c_2} e(t)$$

So,

$$A = \begin{bmatrix} \dfrac{d\,T_1(t)}{dt} \\ \dfrac{d\,T_2(t)}{dt} \end{bmatrix} = A \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix} + B \begin{bmatrix} T_0(t) \\ e(t) \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} \dfrac{-k_{12}}{m_1 c_1} & \dfrac{k_{12}}{m_1 c_1} \\ \dfrac{k_{12}}{m_2 c_2} & \dfrac{-k_{12} - k_{20}}{m_2 c_2} \end{bmatrix}$$

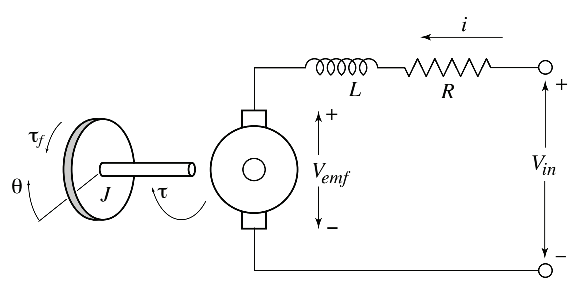
**Exercises:**

1. What is $B$?
2. What is $C$?
3. What is $D$?
4. What would $C$ and $D$ be if we had both $T_1(t)$ and $T_2(t)$ as outputs?
5. Calculate the transfer functions from $e(t)$ and $T_0(t)$ to $T_1(t)$.





**DC Motor connected to a rotational load**




The "back emf" is proportional to motor speed: $V_{emf} = K \dfrac{d\theta}{dt}$.

The motor has a series resistance, $R$, and inductance, $L$, so $V_{in} = V_{emf} + L\dfrac{d\,i}{dt} + R\,i$.

The motor torque is proportional to the motor current: $\tau = K_\tau\, i$.

There is a friction torque, $\tau_f$ opposing the motor and proportional to its speed: $\tau_f = b\,\dfrac{d\theta}{dt}$.

The rotational load has inertia $J$ so the torque balance equation is: $J\,\dfrac{d^2\theta}{dt^2} + \tau_f = \tau$.

**DC Motor connected to a rotational load**

We are interested in the model from the input voltage, $V_{in}$, to the rotor angle, $\theta$.

**Exercise:**

1. How many states are required for this model?
2. Derive a state-space representation.
3. What are the poles of the system?