Solving the state-space differential equation (Laplace approach)

Consider a continuous time state-space representation.

$$\begin{aligned}\frac{dx(t)}{dt} &= A\,x(t) \;+\; B\,u(t),\\ y(t) &= C\,x(t) \;+\; D\,u(t).\end{aligned}$$

To begin, look at the zero-input solution

$$\frac{dx(t)}{dt} = A\,x(t).$$

Taking unilateral Laplace transforms gives,

$$sx(s) - x(0) = A\,x(s) \qquad \text{where} \quad x(0) = x(t) \text{ at } t = 0.$$

This gives,

$$(sI - A)x(s) = x(0), \qquad \text{or} \qquad x(s) = (sI - A)^{-1}x(0).$$

Taking an inverse Laplace transform gives,

$$x(t) \quad = \quad \mathcal{L}^{-1}\big\{(sI - A)^{-1}\big\}\; x(0) \quad = \quad \Phi(t)x(0).$$

$\Phi(t)$, is also known as the *"State Transition Matrix"*.

Solving the state-space differential equation (Laplace approach)

Now consider the case where there is an input.

$$\frac{dx(t)}{dt} = A\,x(t) \;+\; B\,u(t).$$

Taking Laplace transforms gives,

$$sx(s) - x(0) = A\,x(s) \;+\; B\,u(s),$$

which means that,

$$x(s) = (sI - A)^{-1}\,x(0) \;+\; (sI - A)^{-1}\,B\,u(s).$$

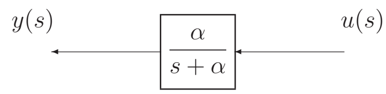
Now the inverse Laplace gives (recall that $\Phi(t)$ is defined as $\mathcal{L}^{-1}\big\{(sI - A)^{-1}\big\}$),

$$x(t) \quad = \quad \underbrace{\Phi(t)\,x(0)}_{\text{zero-input solution}} \quad + \quad \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau.}_{\text{convolution of } \Phi(t) \text{ and } B\,u(t)}$$

We can also use this equation for an arbitrary initial time,

$$x(t) = \Phi(t - t_0)\,x(t_0) \quad + \quad \int_{t_0}^t \Phi(t-\tau)\,B\,u(\tau)\,d\tau.$$

A simple example: A first order system ($\alpha > 0$).



Take the initial conditions to be zero. The differential equation is,

$$\frac{dy(t)}{dt} + \alpha\, y(t) \;=\; \alpha\, u(t).$$

Define the state as, $x(t) = y(t)$, then,

$$\begin{aligned} \frac{dx(t)}{dt} &= -\alpha\, x(t) \;+\; \alpha u(t), \\ y(t) &= x(t) \end{aligned} \qquad \text{or} \qquad \begin{aligned} \frac{dx(t)}{dt} &= \begin{bmatrix}-\alpha\end{bmatrix} x(t) \;+\; \begin{bmatrix}\alpha\end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix}1\end{bmatrix} x(t) \;+\; \begin{bmatrix}0\end{bmatrix} u(t). \end{aligned}$$

So the state-space representation is: $A = -\alpha$, $B = \alpha$, $C = 1$ and $D = 0$.

Example (continued)

Now look at the State Transition Matrix,

$$\begin{aligned} \Phi(t) &= \mathcal{L}^{-1}\big\{(sI - A)^{-1}\big\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}\right\} \\ &= \mathrm{e}^{-\alpha t} \qquad \text{(this looks just like the impulse reponse)} \end{aligned}$$

So we can calculate ,

$$x(t) = \mathrm{e}^{-\alpha t}\, x(0) \;+\; \int_0^t \mathrm{e}^{-\alpha(t-\tau)}\, \alpha\, u(\tau)\, d\tau.$$

Step response: Zero initial condition ($x(0) = 0$).

As $u(t) = 1$ for $t \geq 0$,

$$\begin{aligned} y(t) \;=\; x(t) &= \mathrm{e}^{-\alpha t}\, 0 \;+\; \int_0^t \mathrm{e}^{-\alpha(t-\tau)}\, \alpha\, d\tau, \quad = \quad \mathrm{e}^{-\alpha t} \int_0^t \mathrm{e}^{\alpha\tau}\, \alpha\, d\tau, \\ &= \mathrm{e}^{-\alpha t}\left[\mathrm{e}^{\alpha\tau}\,\big|_{\tau=t} - \mathrm{e}^{\alpha\tau}\,\big|_{\tau=0}\right] \quad = \quad \mathrm{e}^{-\alpha t}\left(\mathrm{e}^{\alpha t} - 1\right) \quad = \quad 1 - \mathrm{e}^{-\alpha t}. \end{aligned}$$

Matrix exponential approach

We will look at an alternative way of deriving $\Phi(t)$ using the *"matrix exponential"*.

Consider the zero-input case, $\dfrac{dx(t)}{dt} \;=\; A\,x(t)$.

Assume that $x(t)$ is smooth and look at an expansion of $x(t)$ about a point $t = t_0$.

$$x(t) \;=\; v_0 \;+\; v_1(t-t_0) \;+\; v_2(t-t_0)^2 \;+\; \cdots$$

Here the v_i are constant vectors of the same size as $x(t)$.

Let's use the differential equation to work out v_i, $i = 0, 1, \ldots$

As the above is supposed to hold for all t, choose $t = t_0$ to get,

$x(t_0) \;=\; v_0$ This gives us the first vector in the expansion.

Matrix exponentials

To get the next term differentiate the expansion to give,

$$\frac{dx(t)}{dt} \;=\; v_1 \;+\; 2v_2(t-t_0) \;+\; 3v_3(t-t_0)^2 \;+ \cdots$$

Again choosing $t = t_0$ gives,

$$\frac{dx(t_0)}{dt} \;=\; v_1.$$

But we know that $\dfrac{dx(t)}{dt} \;=\; A\,x(t)$, so,

$$v_1 = A\,x(t_0).$$

Differentiate again to get,

$$\frac{d^2x(t)}{dt^2} \;=\; 2v_2 \;+\; 6v_3(t-t_0) \;+\; 12v_4(t-t_0)^2 \;+ \cdots$$

Choosing $t = t_0$ gives, $v_2 = \dfrac{1}{2}\dfrac{d^2x(t_0)}{dt^2}$.

But $\dfrac{d^2x(t)}{dt^2} \;=\; \dfrac{dAx(t)}{dt} \;=\; A\dfrac{dx(t)}{dt} \;=\; A^2x(t)$, so

$$v_2 = \frac{A^2}{2}\,x(t_0).$$

We can keep differentiating, substituting $t = t_0$, and solving for the v_i terms.

This eventually gives,

$$\begin{aligned} x(t) &= x(t_0) + A(t-t_0)\,x(t_0) \;+\; \tfrac{A^2}{2}(t-t_0)^2\,x(t_0) \;+\; \tfrac{A^3}{3!}(t-t_0)^3\,x(t_0) \;+\cdots \\ &= \underbrace{\left[I + A(t-t_0) \;+\; \frac{A^2}{2}(t-t_0)^2 \;+\; \frac{A^3}{3!}(t-t_0)^3 \;+\cdots \right]}_{\text{define this as } \mathrm{e}^{A(t-t_0)}} \; x(t_0). \end{aligned}$$

If $t_0 = 0$ (as is usually the case) we have,

$$x(t) \;=\; \mathrm{e}^{At}\,x(0) \qquad \text{and so} \qquad \Phi(t) \;=\; \mathrm{e}^{At}.$$

In MATLAB the command `expm` calculates the matrix exponential.

Caveat emptor. This is not the same as the exponential of the individual elements of a matrix (which is calculated by the MATLAB command: `exp`).

Properties:

$$\mathrm{e}^{A\times 0} = I, \qquad \mathrm{e}^{A(s+t)} = \mathrm{e}^{As}\mathrm{e}^{At}, \qquad \mathrm{e}^{-At}\mathrm{e}^{At} = I, \qquad \frac{d\,\mathrm{e}^{At}}{dt} = A\mathrm{e}^{At}.$$

Matrix exponential approach to solving state-space differential equations.

We can begin by “guessing” a solution of the form,

$$x(t) \;=\; \mathrm{e}^{At}\,v(t), \qquad \text{where } v(t) \text{ is a time-varying vector.}$$

Differentiate this to get,

$$\begin{aligned} \frac{dx(t)}{dt} &= A\mathrm{e}^{At}\,v(t) \;+\; \mathrm{e}^{At}\,\frac{dv(t)}{dt} \\ &= A\,x(t) \;+\; B\,u(t). \qquad \text{(from the differential equation)} \\ &= A\mathrm{e}^{At}\,v(t) \;+\; B\,u(t) \qquad \text{(by substituting for } x(t)\text{)} \end{aligned}$$

and by equating the first & third lines,

$$\begin{aligned} \mathrm{e}^{At}\,\frac{dv(t)}{dt} &= B\,u(t) \\ \frac{dv(t)}{dt} &= \mathrm{e}^{-At}\,Bu(t). \end{aligned}$$

Solve this by integrating to get:

$$v(t) - v(0) \;=\; \int_0^t \mathrm{e}^{-A\tau}\,Bu(\tau)\,d\tau.$$

From before,

$$v(t) - v(0) \;=\; \int_0^t \mathrm{e}^{-A\tau}\, Bu(\tau)\, d\tau.$$

but,

$$x(t) \;=\; \mathrm{e}^{At}\, v(t) \quad \text{so} \quad v(t) \;=\; \mathrm{e}^{-At}\, x(t) \quad \text{and} \quad v(0) \;=\; x(0).$$

Substituting these gives,

$$\mathrm{e}^{-At}\, x(t) - x(0) \;=\; \int_0^t \mathrm{e}^{-A\tau}\, Bu(\tau)\, d\tau.$$

or,

$$\begin{aligned} x(t) \;&=\; \mathrm{e}^{At}\, x(0) \;+\; \mathrm{e}^{At} \int_0^t \mathrm{e}^{-A\tau}\, Bu(\tau)\, d\tau, \\ &=\; \mathrm{e}^{At}\, x(0) \;+\; \int_0^t \mathrm{e}^{A(t-\tau)}\, Bu(\tau)\, d\tau, \end{aligned}$$

This is exactly the same solution as before where $\Phi(t) = \mathrm{e}^{At}$.

Example:

$$P(s) \quad = \quad \frac{(s-1)}{(s+1)(s+2)}.$$

This system has a state-space representation:

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = 0.$$

The state transition matrix is $\Phi(t) = \mathrm{e}^{At} = \mathcal{L}^{-1}\big\{(sI - A)^{-1}\big\}$.

In this example:

$$\mathrm{e}^{At} = \mathcal{L}^{-1}\left\{ \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \right)^{-1} \right\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}^{-1} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s(s+3)+2} \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \begin{bmatrix} \dfrac{2}{s+2} + \dfrac{-1}{s+1} & \dfrac{2}{s+2} + \dfrac{-2}{s+1} \\[2ex] \dfrac{-1}{s+2} + \dfrac{1}{s+1} & \dfrac{-1}{s+2} + \dfrac{2}{s+1} \end{bmatrix} \right\} = \begin{bmatrix} 2\mathrm{e}^{-2t} - \mathrm{e}^{-t} & 2\mathrm{e}^{-2t} - 2\mathrm{e}^{-t} \\ -\mathrm{e}^{-2t} + \mathrm{e}^{-t} & -\mathrm{e}^{-2t} + 2\mathrm{e}^{-t} \end{bmatrix}$$

Transfer functions (continuous time)

Assume that $x(0) = 0$.

$$\begin{aligned} \frac{dx(t)}{dt} &= A\,x(t) \;+\; B\,u(t), \\ y(t) &= C\,x(t) \;+\; D\,u(t). \end{aligned}$$

Turn this into a transfer function simply by taking Laplace transforms and solving for $y(s)/u(s)$.

Doing this gives,

$$s\,x(t) \quad = \quad A\,x(s) \;+\; B\,u(s),$$

which implies that,

$$x(s) \quad = \quad (sI - A)^{-1} B\,u(s).$$

We also have,

$$y(s) = C\,x(s) \;+\; D\,u(s).$$

and substituting for $x(s)$ gives,

$$y(s) = \left[\, C(sI - A)^{-1}B + D \,\right]\; u(s).$$

So the transfer function is:

$$\frac{y(s)}{u(s)} \quad = \quad P(s) \quad = \quad C(sI - A)^{-1}B + D.$$

Example: (trivial)

$$A = -\alpha, \quad B = \alpha, \quad C = 1, \quad D = 0.$$

$$P(s) \quad = \quad C(sI - A)^{-1}B + D \quad = \quad 1(s - (-\alpha))^{-1}\alpha + 0 \quad = \quad \frac{\alpha}{s + \alpha}.$$

Discrete time case:

Perform the same operations with $\mathcal{Z}$-transforms to get,

$$P(z) \quad = \quad C(zI - A)^{-1}B + D.$$

Example: (revisted)

$$P(s) \quad = \quad \frac{(s-1)}{(s+1)(s+2)}.$$

This system has a state-space representation:

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = 0.$$

From before we have,

$$(sI - A)^{-1} = \frac{1}{s(s+3)+2} \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix},$$

so,

$$\begin{aligned} C(sI-A)^{-1}B + D &= \frac{1}{s(s+3)+2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \\ &= \frac{1}{s(s+3)+2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix} \\ &= \frac{s-1}{s^2+3s+2}. \end{aligned}$$

Poles and zeros

If a system, $P(s)$, has a pole at $s = p_i$, then its partial fraction expansion is,

$$P(s) \quad = \quad \frac{a(s)}{b(s)} \quad = \quad \frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_i)\ldots(s-p_n)} \quad = \quad \frac{E_1}{(s-p_1)} + \cdots + \frac{E_i}{(s-p_i)} + \cdots + \frac{E_n}{(s-p_n)}.$$

For the moment assume that p_i is not repeated.

The impulse response will be of the form,

$$p(t) = E_1 e^{p_1 t} + \cdots + E_i e^{p_i t} + \cdots + E_n e^{p_n t}.$$

The zero-input solutions of the corresponding differential equation will have terms of the form,

$$y(t) \quad = \quad k_i e^{p_i t} \; + \; \ldots$$

Look at this idea from a state-space point of view.

State-space point of view:

Consider the zero-input case, $\dfrac{dx(t)}{dt} = A\,x(t)$, and look at solutions of the form,

$$x(t) \quad = \quad e^{p_i t} x(0).$$

Differentiating this gives,

$$\frac{dx(t)}{dt} \quad = \quad p_i e^{p_i t} x_0 \quad = \quad p_i\, x(t).$$

As $\dfrac{dx(t)}{dt} = A\,x(t)$, we have,

$$A\,x(t) \quad = \quad p_i x(t),$$

and taking $t = 0$ gives,

$$A\,x(0) \quad = \quad p_i\, x(0) \qquad \longleftarrow \quad \text{This is an eigenvalue equation.}$$

The eigenvalues of A are the poles of $P(s)$.

The poles (eigenvalues) are also called "natural frequencies" or "modes" of $P(s)$.

Example: (yet again)

$$P(s) \quad = \quad \frac{(s-1)}{(s+1)(s+2)}.$$

This system has a state-space representation:

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = 0.$$

We will sometimes abbreviate this to,

$$P(s) \quad = \quad \left[\begin{array}{cc|c} -3 & -2 & 1 \\ 1 & 0 & 0 \\ \hline 1 & -1 & 0 \end{array}\right]. \qquad \text{Note that the dimensions always make this possible.}$$

Eigenvalue equation:

The eigenvalues of A satisfy,

$$\det(\lambda I - A) = 0.$$

But here we see that this is simply the roots of the denominator,

$$\det \begin{bmatrix} \lambda + 3 & 2 \\ -1 & \lambda \end{bmatrix} \quad = \quad \lambda(\lambda+3) + 2 \quad = \quad \lambda^2 + 3\lambda + 2 \quad = \quad (\lambda+1)(\lambda+2) \quad = \quad 0.$$

Eigenvalues and similarity transforms

We can "diagonalize" most matrices by a transformation,

$$A \quad = \quad V \Lambda V^{-1}, \quad \text{where} \quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{is diagonal.}$$

In the previous example,

$$\underbrace{\begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}}_{A} \quad = \quad \underbrace{\begin{bmatrix} -2 & \sqrt{2}/2 \\ 1 & -\sqrt{2}/2 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ -1 & -2 \end{bmatrix}}_{V^{-1}}$$

The MATLAB command `eig` does this eigenvalue decomposition.

Exercise:

1. Calculate this decomposition by hand.

Similarity transforms

Define a new state, $\xi(t)$, by the linear invertible transform,

$$\xi(t) = V^{-1}\, x(t) \qquad \text{or, equivalently,} \quad x(t) = V\,\xi(t).$$

Recall the usual state-space equations,

$$\begin{aligned} \frac{dx(t)}{dt} &= A\,x(t) \;+\; B\,u(t), \\ y(t) &= C\,x(t) \;+\; D\,u(t), \end{aligned}$$

and substitute for $x(t)$.

$$\begin{aligned} V\frac{d\xi(t)}{dt} &= A\,V\,\xi(t) \;+\; B\,u(t), \\ y(t) &= C\,V\,\xi(t) \;+\; D\,u(t), \end{aligned} \qquad \text{or} \qquad \begin{aligned} \frac{d\xi(t)}{dt} &= V^{-1}\,A\,V\,\xi(t) \;+\; V^{-1}\,B\,u(t), \\ y(t) &= C\,V\,\xi(t) \qquad\; +\; D\,u(t). \end{aligned}$$

Substituting $V^{-1}AV = \Lambda$, gives,

$$\begin{aligned} \frac{d\xi(t)}{dt} &= \Lambda\,\xi(t) \qquad + \; V^{-1}\,B\,u(t), \\ y(t) &= C\,V\,\xi(t) \;+\; D\,u(t). \end{aligned}$$

This is a new (and completely equivalent) state-space description.

Similarity transforms

Any invertible $n \times n$ matrix, V, can be used to transform a matrix,

$$\hat{A} = V\,A\,V^{-1}.$$

The matrix, $\hat{A}$ is *"similar"* to (has the same eigenvalues as) A.

This is called a *"similarity transform"*.

Similarity transformed state-space representations

By defining a new state, $\xi(t) = V^{-1}\,x(t)$,

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad = \quad \left[\begin{array}{c|c} V^{-1}AV & V^{-1}B \\ \hline CV & D \end{array}\right].$$

So there are obviously an infinite number of equivalent state-space representations.

Exercises:

1. Show that the eigenvalues of A are unchanged by a similarity transform.
2. Show that the transfer function is unchanged by a similarity transform of A.

Zeros

If $P(s) \;=\; \dfrac{y(s)}{u(s)}$ has a zero at $s = s_0$, then for all inputs $u(s_0)$ the output $y(s_0) = 0$.

This is equivalent to,

$$0 = C(s_0 I - A)^{-1}B \;+\; D$$

Using a state-space Laplace domain form (at $s = s_0$) we have,

$$\begin{aligned} s_0 x(s_0) &= A\,x(s_0) + B\,u(s_0) \\ 0 &= C\,x(s_0) + D\,u(s_0) \end{aligned}$$

This can be rearranged into matrix form,

$$\begin{bmatrix} (s_0 I - A) & -B \\ C & D \end{bmatrix} \begin{bmatrix} x(s_0) \\ u(s_0) \end{bmatrix} = 0.$$

This means that the matrix,

$$\begin{bmatrix} (s_0 I - A) & -B \\ C & D \end{bmatrix} \quad \text{is singular, or equivalently} \quad \det \begin{bmatrix} (s_0 I - A) & -B \\ C & D \end{bmatrix} = 0.$$

Summary: Continuous time

Poles are given by: $\det(sI - A) = 0$

Zeros are given by: $\det \begin{bmatrix} (sI - A) & -B \\ C & D \end{bmatrix} = 0.$

Summary: Discrete time

Poles are given by: $\det(zI - A) = 0$

Zeros are given by: $\det \begin{bmatrix} (zI - A) & -B \\ C & D \end{bmatrix} = 0.$