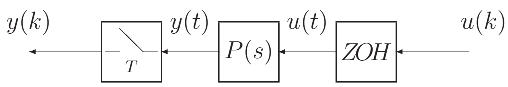
Continuous to discrete transforms in state-space

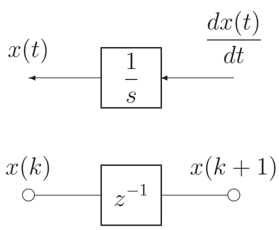
We have several ways of calculating a discrete-time transfer function from a continuous-time one, depending on the application.

ZOH Equivalence



$$P(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{ \frac{P(s)}{s} \right\}.$$

Controller approximation

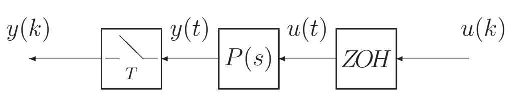


Forward difference: $C(z) = C(s) \,|_{s=\frac{z-1}{T}}$

Backward difference: $C(z) = C(s) \,|_{s=\frac{z-1}{Tz}}$

Tustin/bilinear: $C(z) = C(s) \,|_{s=\frac{2}{T}\,\frac{(z-1)}{(z+1)}}$

ZOH Equivalence



$$P(s) \quad \begin{cases} \dfrac{dx(t)}{dt} & = \; A\,x(t) \; + \; B\,u(t) \\ \quad y(t) & = \; C\,x(t) \; + \; D\,u(t). \end{cases}$$

We would like to get a description of the form,

$$P(z) \quad \begin{cases} x(k+1) & = \; A_d\,x(k) \; + \; B_d\,u(k) \\ \quad y(k) & = \; C_d\,x(k) \; + \; D_d\,u(k). \end{cases}$$

Approach: Solve the state equation over one sample period.

$$x(t) \;=\; \mathrm{e}^{At}\,x(0) \;+\; \int_0^t \mathrm{e}^{A(t-\tau)}\,Bu(\tau)\,d\tau,$$

And over a single sample period (kT to $kT + T$) this is,

$$x(kT+T) \;=\; \mathrm{e}^{AT}\,x(kT) \;+\; \int_{kT}^{kT+T} \mathrm{e}^{A(kT+T-\tau)}\,Bu(\tau)\,d\tau,$$

Key observation

The integration involves $u(\tau)$ from $\tau = kT$ to $\tau = kT + T$.

But $u(\tau)$ is **constant** over this time period. It is the output of a *ZOH*.

So, $u(\tau) = u(k)$ for $kT \leq \tau < kT + T$.

Therefore,

$$x(kT+T) \;=\; \mathrm{e}^{AT}\,x(kT) \;+\; \left[\int_{kT}^{kT+T} \mathrm{e}^{A(kT+T-\tau)}\,B\,d\tau,\right] u(k).$$

By our sampling definitions, $x(t)\,|_{t=kT} = x(k)$, so

$$\begin{aligned} x(k+1) \;&=\; \mathrm{e}^{AT}\,x(k) \;+\; \left[\textstyle\int_{kT}^{kT+T} \mathrm{e}^{A(kT+T-\tau)}\,B\,d\tau,\right] u(k). \\ &=\; A_d\,x(k) \;+\; B_d\,u(k), \end{aligned}$$

$$\text{where,}\quad A_d = \mathrm{e}^{AT}, \quad \text{and} \quad B_d = \int_{kT}^{kT+T} \mathrm{e}^{A(kT+T-\tau)}\,B\,d\tau.$$

To simplify the B_d integral define $\eta = kT + T - \tau$ to get,

$$B_d = \int_0^T \mathrm{e}^{A\eta}\,B\,d\eta.$$

***ZOH* equivalent**

So far we have calculated $x(k+1)$ as a linear function of $x(k)$ and $u(k)$. What about $y(k)$?

$$y(kT) \;=\; C\,x(kT) \;+\; D\,u(kT).$$

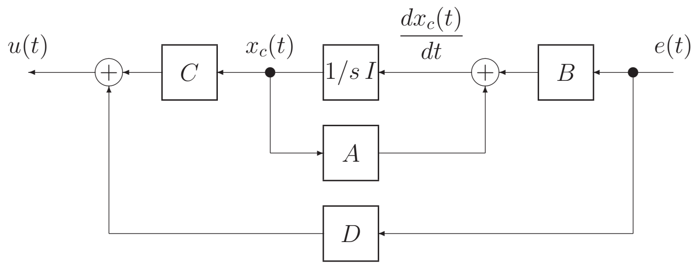
By definition, $y(k) = y(kT)$, and as $u(t)$ is constant over the sample period, $u(k) = u(kT)$.

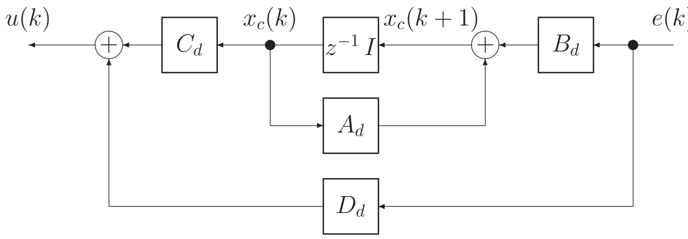
$$y(k) \;=\; C\,x(k) \;+\; D\,u(k).$$

Clearly then, $C_d = C$ and $D_d = D$.

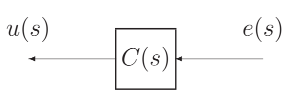
$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \overset{\text{ZOH}}{\Longrightarrow} \quad \left[\begin{array}{c|c} \mathrm{e}^{AT} & \displaystyle\int_0^T \mathrm{e}^{A\eta}\,B\,d\eta \\ \hline C & D \end{array}\right]$$

A_d and B_d are calculated via MATLAB commands `c2d` or `zohequiv`.





Controller (continuous-time)



$$\begin{aligned}\frac{dx_c(t)}{dt} &= A\, x_c(t) \;+\; B\, e(t)\\ u(t) &= C\, x_c(t) \;+\; D\, e(t).\end{aligned}$$

Forward difference approximation $\quad \dfrac{1}{s} \approx \dfrac{T}{z-1} \quad \text{or} \quad s \approx \dfrac{z-1}{T}.$

Take a Laplace transform of the controller equations,

$$\begin{aligned} s\, x_c(s) &= A\, x_c(s) \;+\; B\, e(s)\\ u(s) &= C\, x_c(s) \;+\; D\, e(s),\end{aligned}$$

and substitute, $s = \dfrac{z-1}{T}$, $\quad x(s) = x(z)$, $\quad e(s) = e(z)$ and $\quad u(s) = u(z)$.

This effectively replaces the $1/s$ block with a forward difference approximation to integration, and relabels all of the signals in the diagram as discrete-time signals.

The result of these substitutions is,

$$\begin{aligned} \frac{z-1}{T}\, x_c(z) &= A\, x_c(z) \;+\; B\, e(z) \\ u(z) &= C\, x_c(z) \;+\; D\, e(z). \end{aligned}$$

This is easily rearranged to get,

$$\begin{aligned} (z-1)\, x_c(z) &= AT\, x_c(z) &+& \;BT\, e(z) \\ u(z) &= C\, x_c(z) &+& \;D\, e(z), \end{aligned} \quad \text{or,} \quad \begin{aligned} z\, x_c(z) &= (I + AT)\, x_c(z) &+& \;BT\, e(z) \\ u(z) &= C\, x_c(z) &+& \;D\, e(z). \end{aligned}$$

This is now in discrete-time state-space form,

$$\begin{aligned} x_c(k+1) &= (I + AT)\, x_c(k) \;+\; BT\, e(k) \\ u(k) &= C\, x_c(k) \;+\; D\, e(k). \end{aligned}$$

Clearly then,

$$A_d \;=\; I + AT, \quad B_d \;=\; BT, \quad C_d \;=\; C \quad \text{and } D_d \;=\; D.$$

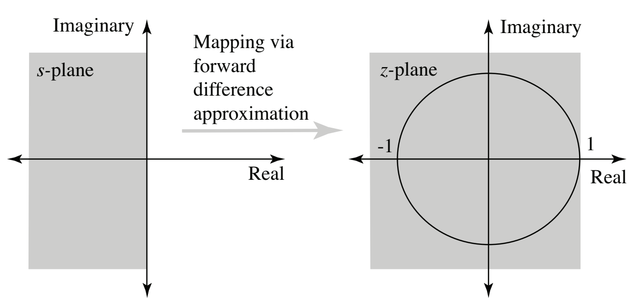
Poles of $C(z)$? Compare the eigenvalues of A_d to A.

$$A_d \;=\; I + AT$$

Exercises: Use the determinant definition of eigenvalues to show:

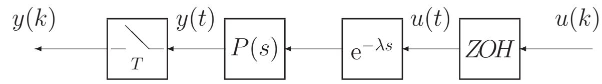
1. multiplying a matrix by a scalar multiplies all of the eigenvalues by the scalar;
2. adding the identity to a matrix adds 1 to all of the eigenvalues.

Eigenvalue/pole mapping result:



Exercise: Repeat this for the backward difference and bilinear approximations.

Delays at the plant input:



This can arise in several ways:

1. The plant has a transport delay at its input. For example a flow delay in a process control system.
2. The delay may actually be introduced by the calculation time of the controller (fraction of a sample period).

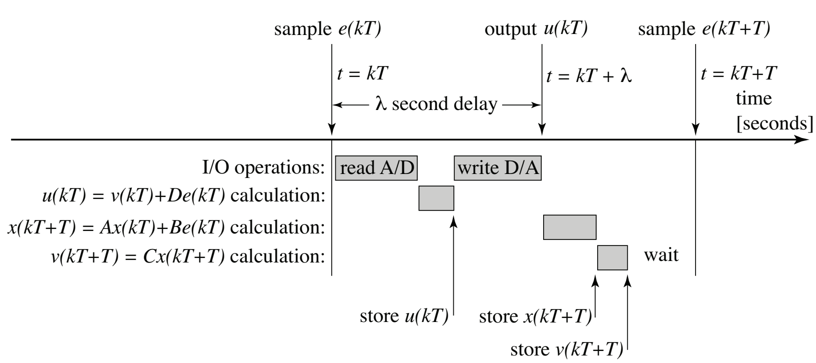
The integer sample period part of any delay easy to handle. If the delay is mT seconds then simply augment the plant with z^{-m}.

The fractional part is harder. . .

Controller timing diagram: Fast sampling with respect to calculation time.

Define an intermediate variable to calculate $u(k)$ as fast as possible.

$$
\begin{aligned}
v(k) &= C\, x_c(k) \\
u(k) &= v(k) \;+\; D\, e(k) \\
x_c(k+1) &= A\, x_c(k) \;+\; B\, e(k)
\end{aligned}
$$



Computation causes a delay (λ seconds) between controller input, $e(k)$, and output, $u(k)$.

Deriving the state-space representation

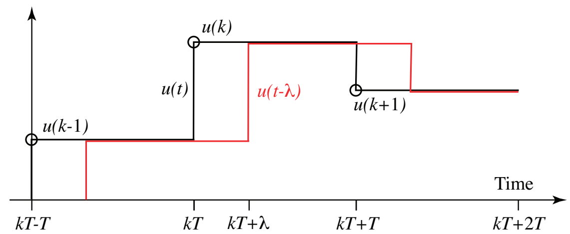
The approach is the same as before, solve the state equation between $t = kT$ and $t = kT + T$.

Because of the λ second delay (note that $\lambda < T$), the state and output equations are now

$$
\begin{aligned}
x(kT+T) &= e^{AT}\, x(kT) \; + \int_{kT}^{kT+T} e^{A(kT+T-\tau)}\, Bu(\tau - \lambda)\, d\tau, \\
y(kT) &= C\, x(kT) \; + \; D\, u(kT - \lambda).
\end{aligned}
$$

Key observation: $u(t)$ now has two constant values over the sample period:

$$
u(t-\lambda) \; = \; \begin{cases} u(k-1) & \text{for } kT \leq t < kT + \lambda, \\ u(k) & \text{for } kT + \lambda \leq t < kT + T \end{cases}
$$



Solving the state equation

We now split the state equation integral into two pieces, corresponding to the times when the input is constant.

$$
\begin{aligned}
x(k+1) &= e^{AT}\, x(k) \; + \int_{kT}^{kT+\lambda} e^{A(kT+T-\tau)} B\, u(\tau-\lambda)\, d\tau \; + \int_{kT+\lambda}^{kT+T} e^{A(kT+T-\eta)} B\, u(\eta - \lambda)\, d\eta \\
&= e^{AT}\, x(k) \; + \left[\int_{kT}^{kT+\lambda} e^{A(kT+T-\tau)} B\, d\tau\right] u(k-1) \; + \left[\int_{kT+\lambda}^{kT+T} e^{A(kT+T-\eta)} B\, d\eta\right] u(k)
\end{aligned}
$$

Define $\xi \; = \; kT - \tau + \lambda$ which means that $d\xi \; = \; -d\tau$ and,

$$
\begin{aligned}
\tau = kT \quad &\Longrightarrow \; \xi = \lambda, \\
\tau = kT + \lambda \quad &\Longrightarrow \; \xi = 0.
\end{aligned}
$$

This takes care of the first integral,

$$
\begin{aligned}
x(k+1) &= e^{AT}\, x(k) \; + \left[\int_{0}^{\lambda} e^{A(T-\lambda+\xi)} B\, d\xi\right] u(k-1) \; + \left[\int_{kT+\lambda}^{kT+T} e^{A(kT+T-\eta)} B\, d\eta\right] u(k) \\
&= e^{AT}\, x(k) \; + \left[e^{A(T-\lambda)} \int_{0}^{\lambda} e^{A\xi} B\, d\xi\right] u(k-1) \; + \left[\int_{kT+\lambda}^{kT+T} e^{A(kT+T-\eta)} B\, d\eta\right] u(k)
\end{aligned}
$$

Solving the state equation

So far we have,

$$x(k+1) \;=\; \mathrm{e}^{AT}\,x(k) \;+\; \left[\mathrm{e}^{A(T-\lambda)}\int_0^{\lambda} \mathrm{e}^{A\xi}B\,d\xi\right]\,u(k-1) \;+\; \left[\int_{kT+\lambda}^{kT+T} \mathrm{e}^{A(kT+T-\eta)}B\,d\eta\right]u(k)$$

Define: $\zeta \;=\; kT+T-\eta$ which means that $d\zeta \;=\; -d\eta$ and now,

$$\begin{aligned} \eta = kT+\lambda \quad &\Longrightarrow \quad \zeta = T-\lambda, \\ \eta = kT+T \quad &\Longrightarrow \quad \zeta = 0. \end{aligned}$$

This takes care of the second integral,

$$x(k+1) \;=\; \mathrm{e}^{AT}\,x(k) \;+\; \left[\mathrm{e}^{A(T-\lambda)}\int_0^{\lambda} \mathrm{e}^{A\xi}B\,d\xi\right]\,u(k-1) \;+\; \left[\int_0^{T-\lambda} \mathrm{e}^{A\zeta}B\,d\zeta\right]u(k)$$

We can now calculate all of the matrix terms, but it is still not quite in state-space form (because $x(k+1)$ depends on both $u(k)$ and $u(k-1)$).

To fix this augment the state with $u(k-1)$.
To do this define,

$$w(k) \;=\; u(k-1) \qquad \text{and so,} \qquad w(k+1) \;=\; u(k) \qquad \longleftarrow \text{this is a new state equation}$$

Putting all the pieces together

The transformed state equation is now,

$$\begin{bmatrix} x(k+1) \\ \\ w(k+1) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{AT} & \mathrm{e}^{A(T-\lambda)}\displaystyle\int_0^{\lambda} \mathrm{e}^{A\xi}B\,d\xi \\ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \\ w(k) \end{bmatrix} + \begin{bmatrix} \displaystyle\int_0^{T-\lambda} \mathrm{e}^{A\zeta}B\,d\zeta \\ \\ I \end{bmatrix} u(k).$$

To get the output equation note that $u(kT-\lambda) \;=\; u(k-1) \;=\; w(k)$, so,

$$y(kT) \quad = \quad C\,x(kT) \;+\; D\,u(kT-\lambda)$$

means that,

$$y(k) \quad = \quad \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \;+\; 0\,u(k).$$

When might we do this?

This adds an additional nu states to our state-space description.

It is only worth doing this when the sampling time is close to the cross over frequency. In this case the delay could have a significant effect and we will need a precise model like this one.