

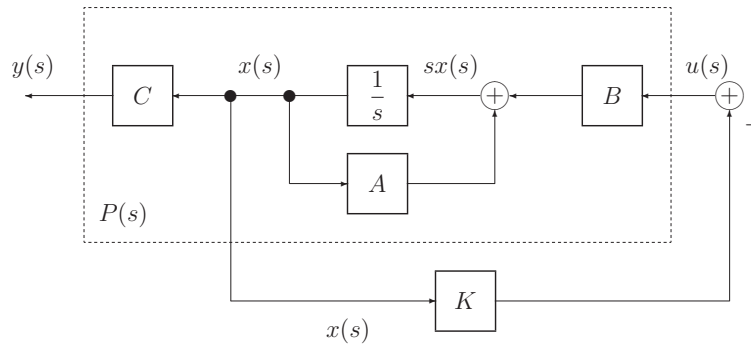
State Feedback

State-space models allow us to use a very powerful design method: *state feedback*.

To begin: assume that we can measure all of the state vector.

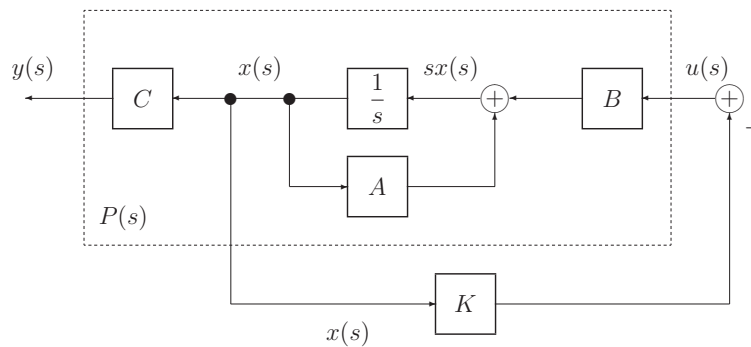
We will address the case where this cannot be done later.

State feedback control law: $u(t) = -Kx(t)$ or, equivalently, $u(s) = -Kx(s)$



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State feedback control law: $u(t) = -Kx(t)$



Salient points:

1. There is no reference input (we will add one later). We are driving the state to zero.
2. We can measure *all* of the states which is often not realistic. We will remove this assumption later by designing an *estimator*.
3. The feedback gain, K , is *static*. This controller has no states.

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Discrete-time state feedback

Given a discrete-time state-space system,

$$x(k+1) = Ax(k) + Bu(k)$$

Recall that the poles of the system are the eigenvalues of A .

State feedback law: $u(k) = -Kx(k)$.

This gives,

$$\begin{aligned} x(k+1) &= Ax(k) - BKx(k) \\ &= \underbrace{(A - BK)}_{\text{closed-loop dynamics}} x(k) \end{aligned}$$

So state feedback changes the pole locations (which is what we expect feedback to do).

The closed-loop poles are given by the eigenvalues of $A - BK$.

Structure of the state feedback control gains:

Examine the feedback gain matrix in more detail,...

$$\begin{aligned} u(k) &= -Kx(k) \\ &= -[K_1 \ K_2 \ \dots \ K_n] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} \end{aligned}$$

Note that K is a static $nu \times nx$ matrix.

We have at least as many control gains in K as states in the system.

We can often use state feedback to place the closed-loop poles *anywhere* we want.

This may take a lot of gain. In other words, large values of K_i .

Calculating K

Start by specifying the desired closed-loop poles. Say they have positions in the z -plane:

$$z = \beta_1, \beta_2, \dots, \beta_n.$$

Note that root-locus arguments, or a knowledge of “nice” pole positions, can often be used to come up with suitable closed-loop pole locations.

The closed-loop pole locations can also be specified as the roots of the polynomial,

$$(z - \beta_1)(z - \beta_2) \cdots (z - \beta_n) = 0.$$

Now, under state feedback, the closed-loop poles are the roots of,

$$\det(zI - (A - BK)) = \det(zI - A + BK) = 0.$$

One approach:

Match coefficients in these two expressions and solve for K_1, K_2, \dots, K_n .

Controllable canonical form

One particular state-space representation (*controllable canonical form*) makes this particularly easy.

For example,

$$P(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}.$$

The controllable canonical form for this is:

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = [b_1 \quad b_2 \quad b_3].$$

Now the closed-loop dynamics, $A - BK$, are given by,

$$\begin{aligned} A - BK &= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [K_1 \quad K_2 \quad K_3], \\ &= \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & -a_3 - K_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \leftarrow \text{easy to see the characteristic polynomial} \end{aligned}$$

Controllable canonical form

The closed-loop matrix is still in controllable canonical form,

$$A - BK = \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & -a_3 - K_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and so,

$$\det(zI - A + BK) = z^3 + (a_1 + K_1)z^2 + (a_2 + K_2)z + (a_3 + K_3).$$

Equating coefficients with the desired closed-loop poles is easy:

Say, the desired closed-loop poles are given by,

$$\begin{aligned} \det(zI - A + BK) &= (z - \beta_1)(z - \beta_2)(z - \beta_3) \\ z^3 + \gamma_1 z^2 + \gamma_2 z + \gamma_3 &= 0, \end{aligned}$$

Then,

$$K_1 = \gamma_1 - a_1, \quad K_2 = \gamma_2 - a_2, \quad K_3 = \gamma_3 - a_3.$$

General case: Ackermann's Formula

Given the state equation (A, B) , and desired closed-loop pole positions (specified by coefficients $\gamma_1, \dots, \gamma_n$), Ackermann's formula calculates K .

First form the "controllability matrix",

$$\mathcal{C} = \underbrace{[B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]}_{n \times n \text{ matrix when } nu = 1}$$

If the desired closed-loop pole positions are given by,

$$z^n + \gamma_1 z^{n-1} + \gamma_2 z^{n-2} + \dots + \gamma_n = 0,$$

then define:

$$\gamma_c(A) := A^n + \gamma_1 A^{n-1} + \dots + \gamma_n I \quad \leftarrow \text{this is an } n \times n \text{ matrix.}$$

Ackermann's formula: $K = [0 \quad \dots \quad 0 \quad 1] \mathcal{C}^{-1} \gamma_c(A)$

In MATLAB, K can be calculated by the command: `place`.

Controllability

Note that Ackermann's formula,

$$K = [0 \ \cdots \ 0 \ 1] \mathcal{C}^{-1} \gamma_c(A),$$

will not work if \mathcal{C}^{-1} does not exist.

Controllability matrix:

$$\mathcal{C} = [B \ AB \ A^2B \ \cdots \ A^{n-1}B].$$

A system is called "*controllable*" if and only if \mathcal{C} is invertible (equivalently, $\text{rank}(\mathcal{C}) = n$).

Formal definition:

A system is *controllable* (or *state controllable*) if and only if for any given initial state, $x(t_0) = x_0$, and any given final state, x_1 , specified at an arbitrary future time, $t_1 > t_0$, there exists a control input $u(t)$ that will drive the system from $x(t_0) = x_0$ to $x(t_1) = x_1$.

Controllability under similarity transforms

What is the effect of a similarity transform, $x(k) = V \xi(k)$.

This gives,

$$\xi(k+1) = V^{-1}AV \xi(k) + V^{-1}B u(k),$$

and the controllability matrix is now,

$$\begin{aligned} \mathcal{C}_\xi &= [(V^{-1}B) \ (V^{-1}AVV^{-1}B) \ (V^{-1}AVV^{-1}AVV^{-1}B) \ \cdots] \\ &= V^{-1} [B \ AB \ A^2B \ \cdots \ A^{n-1}B] \\ &= V^{-1}\mathcal{C}. \end{aligned}$$

Because V^{-1} is invertible, $\text{rank}(\mathcal{C}_\xi) = \text{rank}(\mathcal{C})$.

In other words controllability is not affected by a similarity transform. $\mathcal{C}_\xi \neq \mathcal{C}$, but if one is invertible, so is the other.

We can transform (via similarity) the system into controllable canonical form iff the system is controllable.

Note that controllability applies to the state of a system. It doesn't make sense in terms of transfer functions (which can always be expressed as controllable canonical forms).

Controllability as a design tool

Consider a two-input, two-output system,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} T^2/2 & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

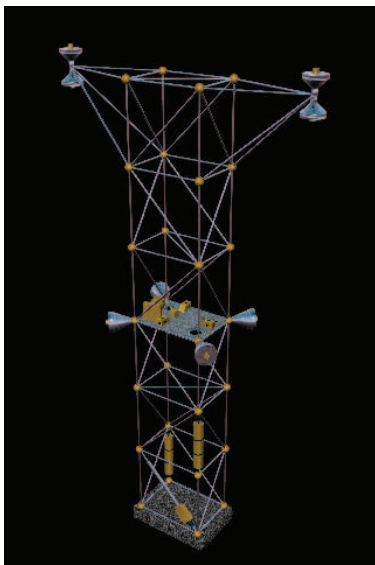
$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},$$

where T is the sample period.

This system is unstable (where are the poles?)

Suppose that implementing $u_1(k)$ costs \$50 and implementing $u_2(k)$ costs \$100. What is the minimum cost to stabilize the system: \$50, \$100 or \$150?

Controllability can answer questions like this and is useful when actually building systems to be controlled.

Another example: Flexible space structures

Large space telescopes:

- Flexible modes must be suppressed
- Limited number of piezoelectric actuators
- Many locations for placing actuators
- Which locations give the most controllable structure

A simple pole placement example

$$P(s) = \frac{-1}{s-1} \quad \leftarrow \text{an unstable system}$$

The ZOH equivalent of this system is,

$$P(z) = \frac{1-e^T}{z-e^T} \quad \leftarrow \text{pole at } z = e^T \text{ (outside unit circle)}$$

One particular state-space representation is,

$$\begin{aligned} x(k+1) &= e^T x(k) + (1-e^T)u(k) \\ y(k) &= 1x(k) \end{aligned}$$

In this case we are measuring the entire state (1 pole = 1 state), and so state feedback is the same as output feedback.

Pole placement: Let's try $\beta = 0$ (deadbeat control).

The desired closed-loop characteristic polynomial is simply $\gamma_c(z) = z = 0$.

Example continued

The closed-loop state-space system with $u(k) = -Kx(k)$ is,

$$\begin{aligned} x(k+1) &= (A - BK)x(k) \\ &= (e^T - (1 - e^T)K)x(k). \end{aligned}$$

And to get the only pole at zero means that $A - BK = 0$.

$$e^T - (1 - e^T)K = 0 \quad \text{which implies that} \quad K = \frac{e^T}{(1 - e^T)}.$$

Controllability?

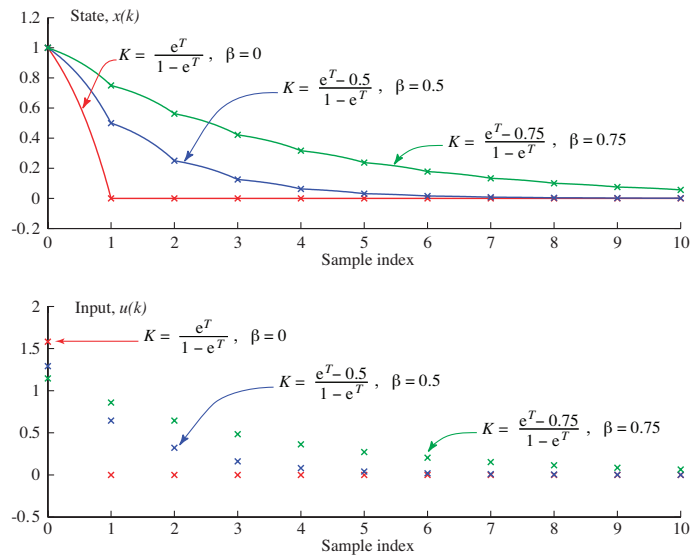
$$C = [B] = 1 - e^T \neq 0, \quad \text{so it is controllable.}$$

Other pole positions

$$\text{For } \beta = 0.5, \quad K = \frac{(e^T - 0.5)}{(1 - e^T)} \text{ and for } \beta = 0.75, \quad K = \frac{(e^T - 0.75)}{(1 - e^T)}.$$

Simulate these choices, with intersample behavior.

Pole position versus actuator effort:



“Nice” intersample behavior is due to the simplicity of the system (1 state).

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State feedback: picking good closed-loop pole locations

Choosing the closed-loop pole locations can be difficult.

You know the general good locations (in z -plane and s -plane). But,...

1. Look at the transient responses of the system and identify the various pole locations. Change the ones that you don't like.
2. The further that you move the poles the greater the gain that you will require. This can lead to cost and/or saturation problems. (I.e. move them as little as possible).
3. Take an “optimal control” course to find out the best tradeoff between fast response and input signal size. This can be viewed as finding K which minimizes the following “cost”:

$$\sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k).$$

where Q and R are positive definite weighting matrices for the state error and the control action respectively. This is known as a Linear Quadratic Regulator (LQR) problem. ECE 230a and ECE 147c cover this material.

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