Reduced order estimation

In many cases we have a partial measurement of the state, and need only estimate the remaining states.

This allows us to use a reduced order estimator. If we can measure m of the nx states, then we need only estimate the $nx - m$ remaining states. What is the resulting controller order?

This allows us to implement simpler controllers.

Example: Inverted pendulum.

The states are:

p	cart position	measured
v	cart velocity	not measured
θ	pendulum angle	measured
ω	pendulum angular velocity	not measured

Use a reduced order estimator to estimate cart velocity, v, and pendulum angular velocity, ω.

The state feedback is then,

$$u(k) = K \begin{bmatrix} p(k) \\ \hat{v}(k) \\ \theta(k) \\ \hat{\omega}(k) \end{bmatrix} \begin{matrix} \\ \longleftarrow \text{ estimated} \\ \\ \longleftarrow \text{ estimated} \end{matrix}$$

How does this differ from simply estimating "rate of change" on each of the two measurements?

Details

To work out the details divide the states into two groups,

$x_a(z)$ Measured states,

$x_b(z)$ Unmeasured states (to be estimated).

with an associated state-space representation,

$$\begin{bmatrix} x_a(k+1) \\ x_b(k+1) \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix}$$

Basic idea: Rearrange unmeasured state equation to make it look like a standard estimation problem.

Examine the unmeasured state update equation,

$$x_b(k+1) = A_{bb}\, x_b(k) + \underbrace{A_{ba}\, x_a(k) + B_b\, u(k)}_{\text{"known" input } =: w(k)}$$

Details (continued)

We can rearrange the $x_a(k+1)$ equation to get something like a standard measurement equation.

$$\underbrace{x_a(k+1) - A_{aa}\, x(k) - B_a\, u(k)}_{\text{"known" measurement } =: v(k)} = A_{ab}\, x_b(k)$$

This gives a smaller state-space system involving only the unmeasured states,

$$x_b(k+1) = A_{bb}\, x_b(k) + w(k)$$

$$v(k) = A_{ab}\, x_b(k)$$

Note that this looks like a standard system with the substitutions,

$$A \longleftarrow A_{bb}, \qquad C \longleftarrow A_{ab}$$

Ackermann's equation lets us design L_r to place the poles of: $A_{bb} - L_r\, A_{ab}$.

Estimator equations

Implementing this estimator gives,

$$\begin{aligned}\hat{x}_b(k+1) \;&=\; A_{bb}\,\hat{x}_b(k) \,+\, A_{ba}\,x_a(k) \,+\, B_b\,u(k) \\ &\quad + L_r\left[x_a(k+1) \,-\, A_{aa}\,x_a(k) \,-\, B_a\,u(k) \,-\, A_{ab}\,\hat{x}_b(k)\right].\end{aligned}$$

Estimator error dynamics

$$\tilde{x}_b(k) \;=\; x_b(k) \,-\, \hat{x}_b(k),$$

so,

$$\begin{aligned}\tilde{x}_b(k+1) \;&=\; x_b(k+1) \,-\, \hat{x}_b(k+1), \\ &=\; A_{ba}\,x_a(k) \,+\, A_{bb}\,x_b(k) \,+\, B_b\,u(k) - A_{bb}\,\hat{x}_b(k) \,-\, A_{ba}\,x_a(k) \,-\, B_b\,u(k) \\ &\quad - L_r\,x_a(k+1) \,+\, L_rA_{aa}\,x_a(k) \,+\, L_rB_a\,u(k) \,+\, L_rA_{ab}\,\hat{x}_b(k), \\ &=\; A_{bb}\,x_b(k) \,-\, A_{bb}\,\hat{x}_b(k) \\ &\quad - L_r\,x_a(k+1) \,+\, L_rA_{aa}\,x_a(k) \,+\, L_rB_a\,u(k) \,+\, L_rA_{ab}\,\hat{x}_b(k).\end{aligned}$$

Estimator error dynamics

$$\begin{aligned}\tilde{x}_b(k+1) \;&=\; A_{bb}\,x_b(k) \,-\, A_{bb}\,\hat{x}_b(k) \,-\, L_r\,x_a(k+1) \\ &\quad + L_rA_{aa}\,x_a(k) \,+\, L_rB_a\,u(k) \,+\, L_rA_{ab}\,\hat{x}_b(k).\end{aligned}$$

Recall that we know how to calculate $x_a(k+1)$,

$$x_a(k+1) \;=\; A_{aa}\,x_a(k) \,+\, A_{ab}\,x_b(k) \,+\, B_a\,u(k),$$

and substituting this gives,

$$\begin{aligned}\tilde{x}_b(k+1) \;&=\; A_{bb}\,x_b(k) \,-\, A_{bb}\,\hat{x}_b(k) \\ &\quad - L_r\left[A_{aa}\,x_a(k) \,+\, A_{ab}\,x_b(k) \,+\, B_a\,u(k)\right] \\ &\quad + L_rA_{aa}\,x_a(k) \,+\, L_rB_a\,u(k) \,+\, L_rA_{ab}\,\hat{x}_b(k) \\ &=\; (A_{bb} \,-\, L_rA_{ab})\;\tilde{x}_b(k).\end{aligned}$$

These are the expected error dynamics for the unmeasured state.

Implementation

There is a problem; the "measurement", $v(k)$, has an $x_a(k+1)$ term in it. So it doesn't appear to be causal.

This can be solved by defining a new state,

$$x_c(k) \;:=\; \hat{x}_b(k) \;-\; L_r\, x_a(k).$$

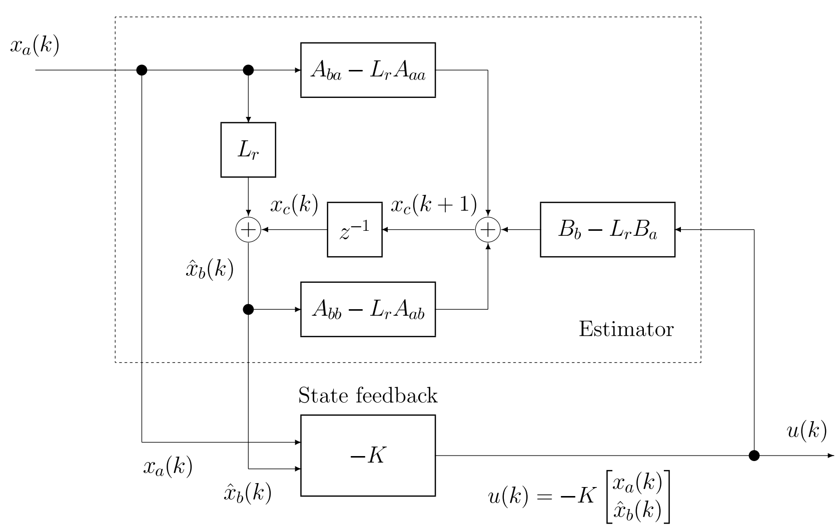
This gives an update equation,

$$\begin{aligned}
x_c(k+1) \;&=\; \boxed{\hat{x}_b(k+1)} \;-\; L_r\, x_a(k+1) \qquad \longleftarrow \text{this is where we will remove } x_a(k+1) \\
&=\; A_{bb}\,\hat{x}_b(k) + A_{ba}\, x_a(k) + B_b\, u(k) \\
&\qquad + L_r\, x_a(k+1) \;-\; L_r A_{aa}\, x_a(k) - L_r B_a\, u(k) - L_r A_{ab}\,\hat{x}_b(k) - L_r\, x_a(k+1) \\
&=\; (A_{bb} \;-\; L_r A_{ab})\;\hat{x}_b(k) \\
&\qquad + (A_{ba} \;-\; L_r A_{aa})\; x_a(k) + (B_b \;-\; L_r B_a)\; u(k).
\end{aligned}$$

Note that $\hat{x}_b(k)$ is easily reconstructed from $x_c(k)$,

$$\hat{x}_b(k) \;=\; x_c(k) \;+\; L_r\, x_a(k). \qquad \longleftarrow \text{substitute to get a state-space system for } x_c(k+1)$$

Implementation



Why estimate when we can measure?

What are the benefits of estimating states when those same states are available for measurement?

Inverted pendulum example

Measured states: p (cart position), θ (pendulum angle)

Options:

1. Estimate v (cart velocity) and ω (pendulum angular velocity) with a differentiator:
$$\hat{v}(k) = \frac{p(k) - p(k-1)}{T}, \qquad \hat{\omega}(k) = \frac{\theta(k) - \theta(k-1)}{T},$$
State feedback uses: $\begin{bmatrix} p(k) & \hat{v}(k) & \theta(k) & \hat{\omega}(k) \end{bmatrix}$
2. Reduced order estimator for $\hat{v}(k)$ and $\hat{\omega}(k)$.
State feedback uses: $\begin{bmatrix} p(k) & \hat{v}(k) & \theta(k) & \hat{\omega}(k) \end{bmatrix}$
3. Full order estimation for all states.
State feedback uses: $\begin{bmatrix} \hat{p}(k) & \hat{v}(k) & \hat{\theta}(k) & \hat{\omega}(k) \end{bmatrix}$

Tradeoffs

Option 1:
- \+ Simple, easy to understand, and easy to debug.
- − The differentiation approximation is very sensitive to high frequency noise. (We could add a low pass filter).
- − The variables p and θ are not independent. They are related by $x(k+1) = Ax(k) + Bu(k)$ and this information is ignored.

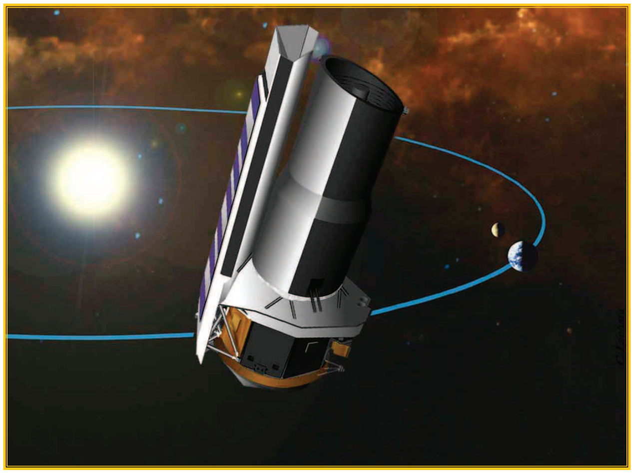
Option 2:
- \+ Low order. Only estimate those states that are needed.
- \+ Optimal estimation of v and ω in the presence of noise on p and θ.
- \+ Dynamic relationship between v and θ taken into account in estimation.
- − Noise on the p and θ variables is not filtered (We could add a low pass noise filter).

Option 3:
- \+ Optimal estimation of all states in the presence of noise on p and θ.
- \+ Dynamic relationship between all states taken into account in estimation.
- − Potentially higher order.

Unless controller order is really critical in the application a full order estimator is preferred.

Estimation application: Spitzer space telescope (SIRTF)

In most earth orbiting and interplanetary missions the estimation is by far the most difficult part of the control problem.

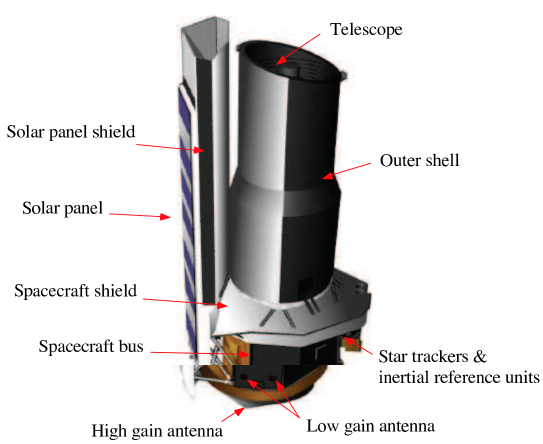


Telescope features:

- Infrared sensing.
- Heliocentric earth trailing orbit (0.12 AU/year drift from Earth)
- Precision pointing (arcsecond)
- Cooled to 5.5K
- Field of view: 32 arcminutes

Credits: David Bayard (JPL)

Pointing control hardware

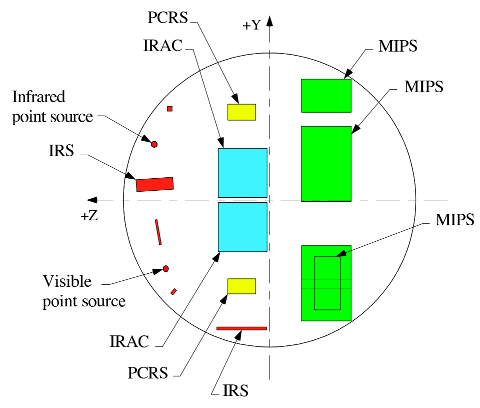


Components:

- Gyro (2)
- Star tracker (2)
- Reaction wheels (4)
- Fine Sun Sensor (2)
- Coarse sun sensor (3)
- PCRS sensor (2)
- Cold gas thrusters (12)

Pointing control hardware

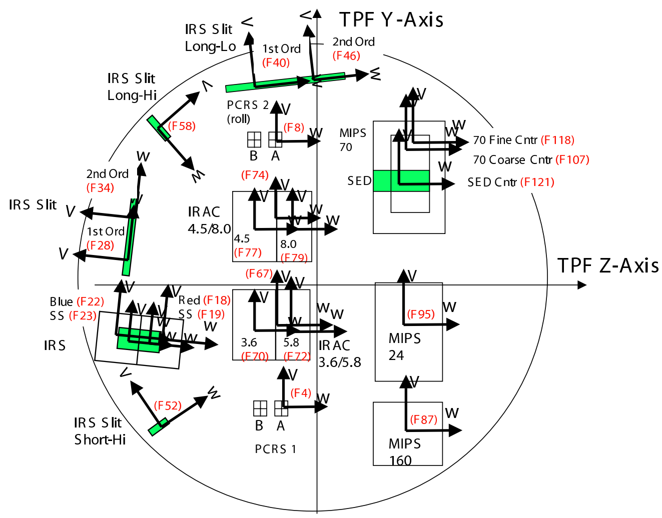
PCRS sensor: optical sensing in the telescope's field-of-view



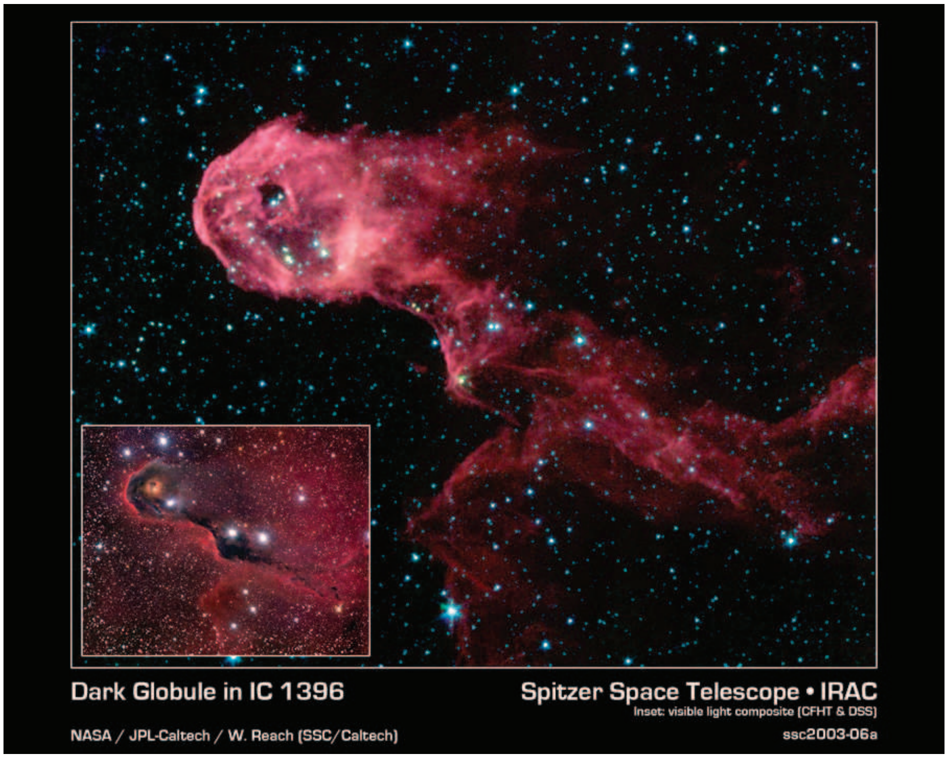
Star trackers

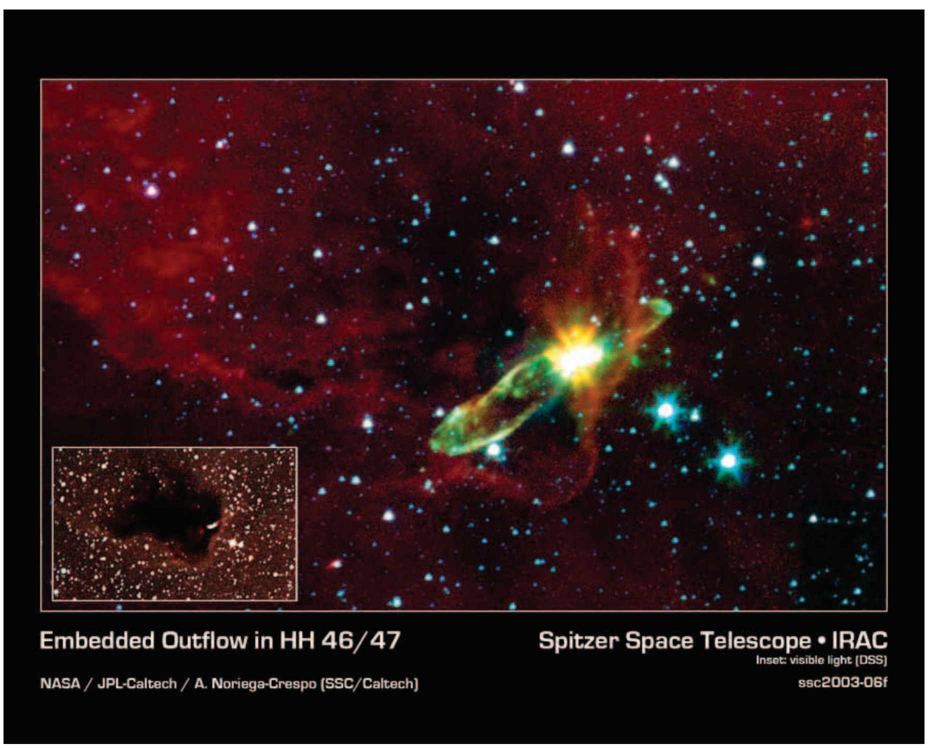
Reference frames

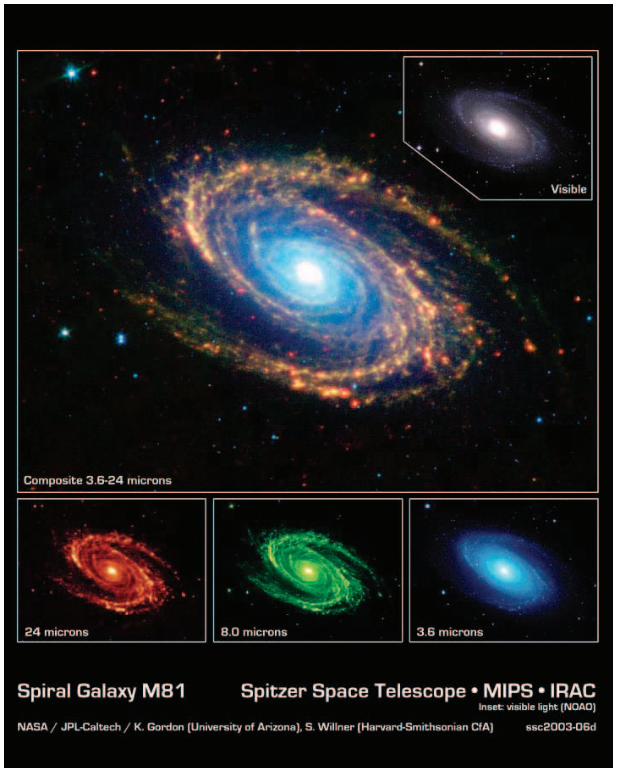
All of the instruments have to be located with respect to the pointing control system.



Filter	**# states**	**Description**	**Ops**	**Update frequency**
Fast Observer	6	Attitude observer 3 attitude states 3 gyro states	flight	2 Hz tracker 10 Hz gyro
STA to PCRS	6	Tracker to telescope alignment 3 short term drift 3 long term drift	flight	8 hours
GCF	18	Gyro calibration 3 scale factors 6 misalignments 3 absolute scale factors 3 gyro bias 3 attitude	flight	Calibrate for 1.5hrs every 4th day Calibrate gyro bias on whenever on inertial hold
PRI	11	Pointing ready indicator 4 two-axis rigid body ($\times 2$) 7 controller states	flight	Every slew controlled using attitude controller
IPF	37	Instrument pointing frame 37 pointing alignments	ground	Several times during in-orbit checkout
PAC	7	Pointing alignment & calibration 6 same as STA-to-PCRS 1 angle between PCRS units	ground	Multiple times during in-orbit checkout


Dark Globule in IC 1396
Spitzer Space Telescope • IRAC
Inset: visible light composite (CFHT & DSS)
NASA / JPL-Caltech / W. Reach (SSC/Caltech)
ssc2003-06a


Embedded Outflow in HH 46/47
Spitzer Space Telescope • IRAC
Inset: visible light (DSS)
NASA / JPL-Caltech / A. Noriega-Crespo (SSC/Caltech)
ssc2003-06f



Optimal estimation: Kalman filtering

Plant inputs: known actuation (u) and unknown disturbance (d) of known variance.

Plant measurements: output (y) plus noise (n) of known variance.

Kalman filter: estimates the state (x) with minimum variance.

The Kalman filter uses the optimal combination of measurement and propagated model to update the estimate.

Applications:

Flight control systems.

GPS navigation, including turn-by-turn navigators.

Economics.

Interplanetary spacecraft guidance, navigation and control.