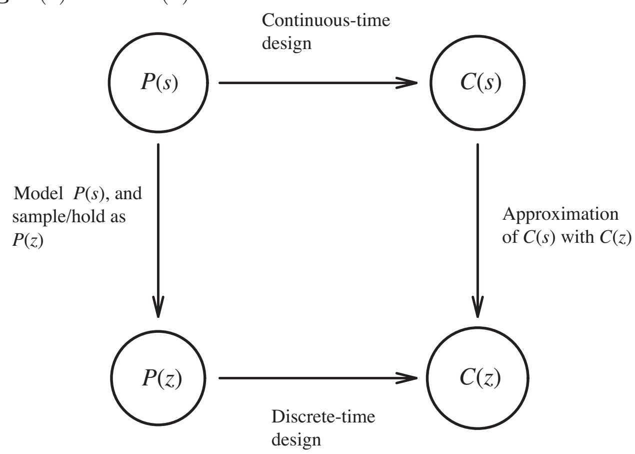
Approximating $C(s)$ with $C(z)$



- Design a continuous-time controller, $C(s)$, for $P(s)$.
- Approximate $C(s)$ with a discrete-time controller, $C(z)$.

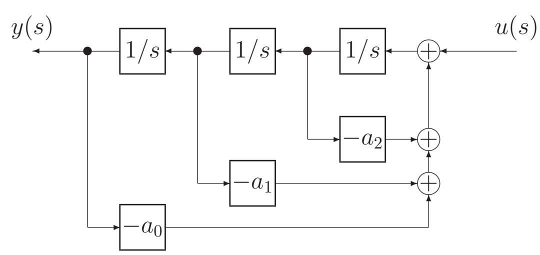
(Franklin & Powell refer to this procedure as "emulation.")

Approach:

A transfer function, $C(s)$, can be realised with integrators, gains, and summation blocks.

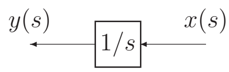
$$C(s) = \frac{y(s)}{u(s)} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}.$$

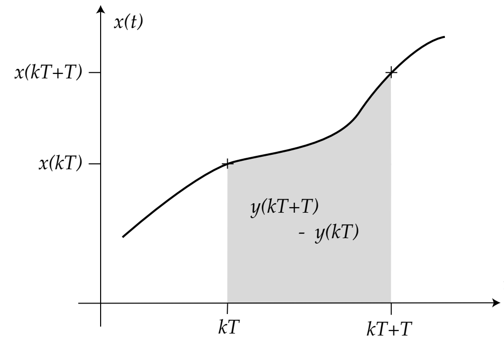
is equivalent to:



Now replace the integrators ($1/s$ blocks) with a discrete-time approximation to integration.

Integration:





$$y(t) = y(0) + \int_0^t x(\tau)\, d\tau,$$

The output, $y(t)$, over a single sample period of T seconds, is given by

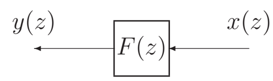
$$y(kT+T) = y(kT) + \int_{kT}^{kT+T} x(\tau)\, d\tau.$$

Objective:

Find a discrete-time approximation, $F(z)$, to the input-output relationship of the integrator.

Find $F(z) \approx 1/s$, then, $s \approx F^{-1}(z)$,

and $C(z) = C(s)\mid_{s=F^{-1}(z)}$.



Forward difference approximation:

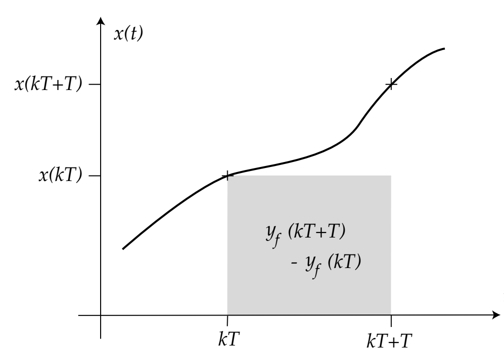
$$y_f(kT+T) = y_f(kT) + Tx(kT).$$

By taking z-transforms,

$$zy_f(z) = y_f(z) + Tx(z),$$

or,

$$\frac{y_f(z)}{x(z)} = \frac{T}{z-1}.$$



So, the approximation is: $\frac{1}{s} \approx \frac{T}{z-1}.$

This is equivalent to the substitution: $s = \frac{z-1}{T}.$

This approximation is also known as an Euler approximation.

Backward difference approximation:

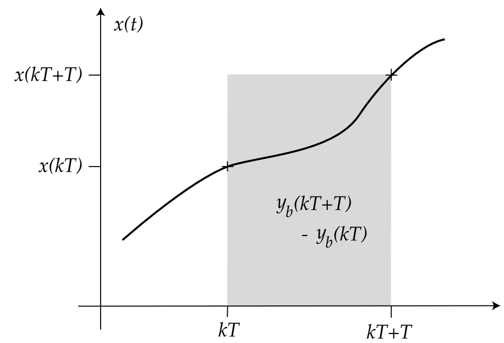
$$y_b(kT+T) = y_b(kT) + Tx(kT+T).$$

In the z-domain this gives,

$$zy_b(z) = y_b(z) + zTx(z),$$

or, equivalently,

$$\frac{y_b(z)}{x(z)} = \frac{Tz}{z-1}.$$



So the approximation is:

$$\frac{1}{s} \approx \frac{Tz}{z-1},$$

which is equivalent to the substitution: $\quad s = \dfrac{z-1}{Tz}.$

Trapezoidal approximation:

$$y_{bl}(kT+T) = y_{bl}(kT) + Tx(kT) + (x(kT+T) - x(kT))T/2.$$

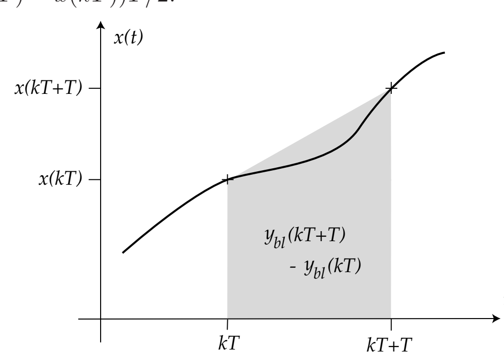
Taking z-transforms,

$$zy_{bl}(z) = y_{bl}(z) + Tx(z) + \frac{T}{2}(z-1)x(z),$$

which gives,

$$\frac{y_{bl}(z)}{x(z)} = \frac{T}{2}\frac{z+1}{z-1}.$$

So the approximation is: $\quad \dfrac{1}{s} \approx \dfrac{T}{2}\dfrac{z+1}{z-1}.$



The substitution is therefore, $\quad s = \dfrac{2}{T}\dfrac{z-1}{z+1}.$

This approximation is also known as:

- Bilinear approximation (based on the mathematical form).
- Tustin approximation (from the British engineer who first used it for this purpose).

Properties:

Controller order:

The forward, backward and trapezoidal approximations all preserve the order of the controller.

If $C(s)$ is an nth order transfer function, the $C(z)$ is also nth order with any of these approximations.

It is possible to derive higher order approximations to integration (quadratic or higher order polynomial fits). These will make the order of $C(z)$ greater than $C(s)$.

Stability:

Two issues:

- **Controller stability:** If $C(s)$ is stable, is $C(z)$ stable?
- **Closed-loop stability:** If $\dfrac{1}{1+P(s)C(s)}$ is stable, is $\dfrac{1}{1+P(z)C(z)}$ stable?

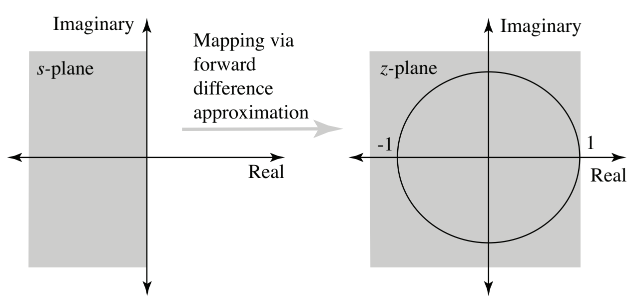
To investigate controller stability we have to look more closely at how the approximations map the s-plane to the z-plane.

Controller stability:

Forward difference/Euler approximation:

$$s = \frac{z-1}{T}.$$

This maps the left half s-plane onto the region shown.



This maps to more than just the unit disk.

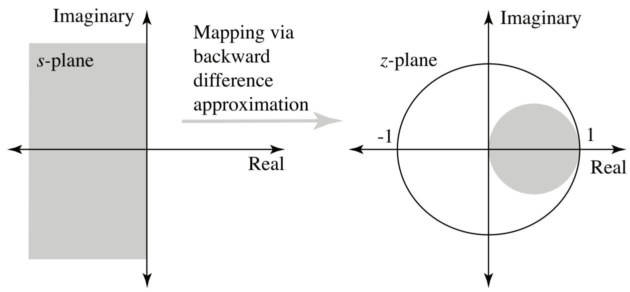
Controllers, $C(s)$, with high frequency or lightly damped poles will give **unstable** $C(z)$.

Controller stability:

Backward difference approximation:

$$s = \frac{z-1}{Tz},$$

This maps the left half s-plane onto the region shown.



This maps to the inside of the unit disk. So stable $C(s)$ imples **stable** $C(z)$.

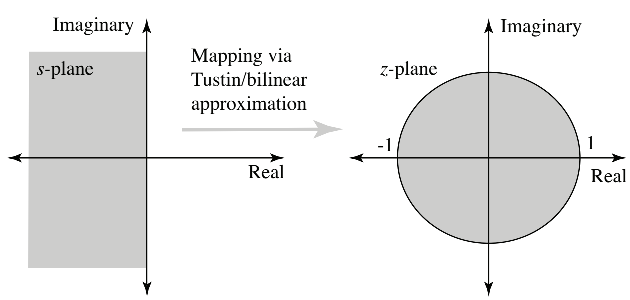
$C(z)$ cannot have lightly damped poles, even if $C(s)$ had lightly damped poles.

Controller stability:

Trapezoidal/Bilinear/Tustin approximation:

$$s = \frac{2}{T}\frac{z-1}{z+1},$$

This maps the left half s-plane onto the region shown.



This maps to the entire right-half plane exactly onto the unit disk.

So $C(s)$ is stable $\iff$ $C(z)$ is **stable**.

This is why this approximation is the most commonly used.

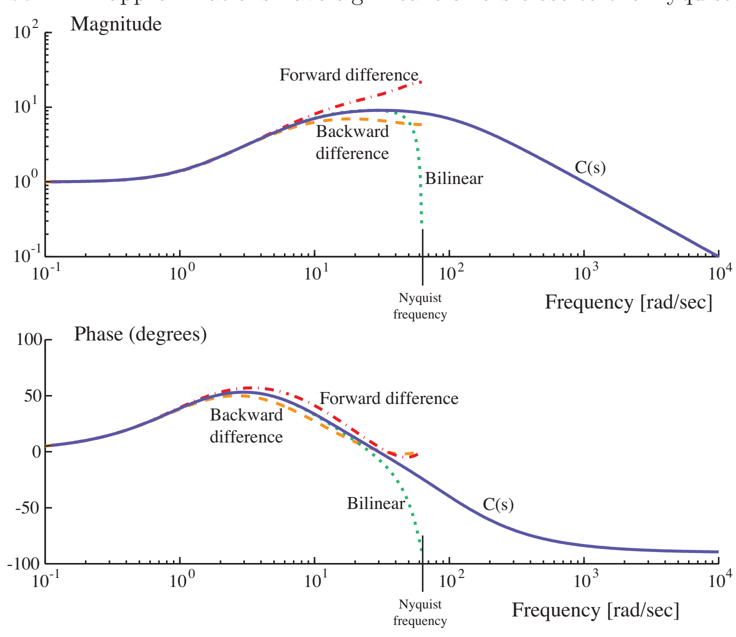
A Comparison

Consider the controller: $C(s) = \dfrac{(s+1)}{(0.1s+1)(0.01s+1)}$.

A lead-lag controller producing the maximum phase lead around 30 rad/sec. ($\approx$ 4.8 Hz).

Using a sample period of $T = 0.05$ second gives a Nyquist frequency of 10 Hz.

A Comparison: All approximations have significant errors close to the Nyquist frequency.



Frequency distortion: Bilinear approximation

Bilinear approximation maps **all** continuous frequencies (ω) from 0 to $j\infty$ to discrete frequencies ($e^{j\Omega T}$) with Ω from 0 to π/T. In particular, $s = j\infty$ maps to $z = e^{j\pi} = -1$.

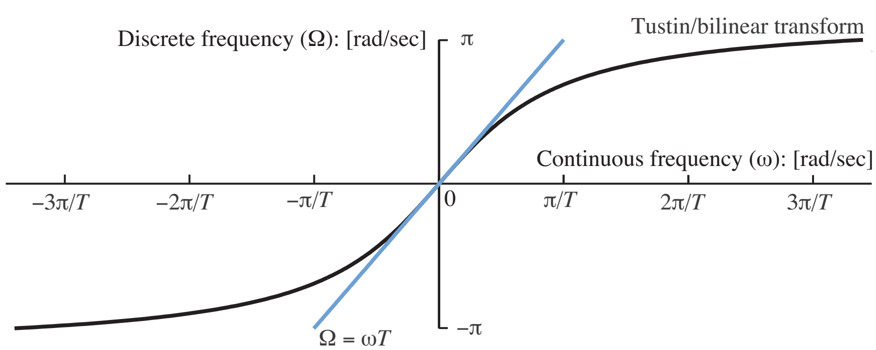
Sampling would map frequencies via $\omega = \Omega$, so $z = -1$ would correspond to a continuous frequency $\omega = j\pi/T$.

Substituting $s = j\omega$ and $z = e^{j\Omega T}$ into $s = \frac{2}{T}\frac{z-1}{z+1}$, gives,

$$\begin{aligned} j\omega &= \frac{2}{T}\frac{(1-e^{-j\Omega T})}{1+e^{-j\Omega T}} \\ &= \frac{2}{T}\frac{j\sin(\Omega T/2)}{\cos(\Omega T/2)} \\ &= \frac{2}{T} j\tan(\Omega T/2), \end{aligned}$$

which implies that the distortion is given by $\quad \Omega = \frac{2}{T}\tan^{-1}(\omega T/2).$

Frequency distortion (Bilinear approximation) $\quad \Omega = \frac{2}{T}\tan^{-1}(\omega T/2).$



The line $\Omega = \omega T$ is the equivalent sampled frequency mapping.

Reducing the distortion: prewarping

The transformation $s = \dfrac{\alpha(z-1)}{(z+1)}$, maps $\mathrm{Re}\{s\} < 0$ to $|z| < 1$.

α is a degree of freedom that can be exploited to modify the frequency distortion.

Prewarping:

Select α to make $C(j\omega_0) = C_z\left(\mathrm{e}^{j\omega_0 T}\right)$.

This makes $C(s) = C_z(z)$ at DC and at $s = j\omega_0$ (ω_0 is the *prewarping frequency*).

To solve for α,

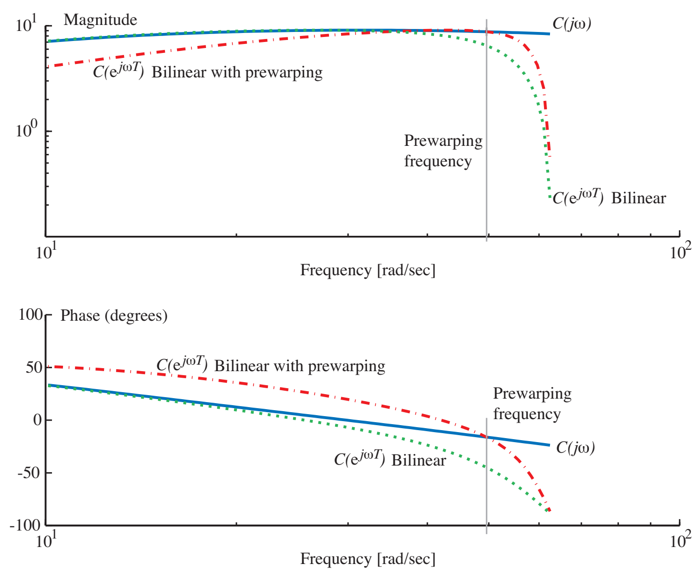
$$j\omega_0 \quad = \quad \frac{\alpha(\mathrm{e}^{j\omega_0 T} - 1)}{(\mathrm{e}^{j\omega_0 T} + 1)} \quad = \quad j\alpha \tan(\omega_0 T/2),$$

which implies that

$$\alpha = \frac{\omega_0}{\tan(\omega_0 T/2)}.$$

Example revisited Choose a prewarping frequency: $\omega_0 = 50$ rad/sec.

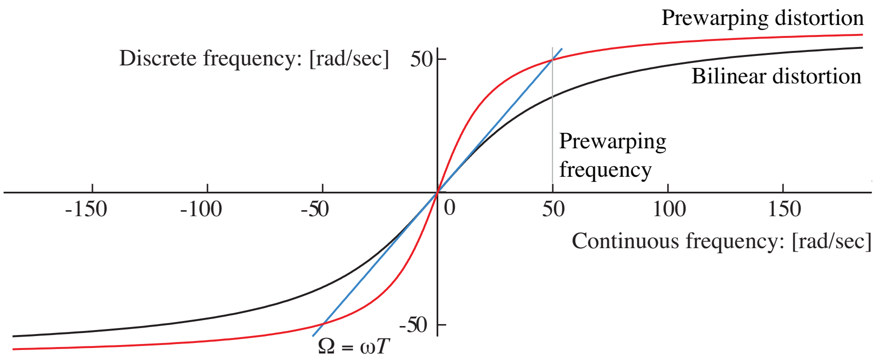
Prewarped bilinear/Tustin: $C_z(z) = C(s)\,\big|_{s=\alpha\frac{z-1}{z+1}}$ which gives $C(j50) = C_z(\mathrm{e}^{j50T})$.



Example revisited

Frequency distortion (Bilinear): $\Omega = \frac{2}{T}\tan^{-1}(\omega T/2)$.

Frequency distortion (Bilinear with prewarping): $\Omega = \frac{2}{T}\tan^{-1}(\omega/\alpha)$



Choosing a prewarping frequency

The prewarping frequency must be in the range: $0 < \omega_0 < \pi/T$.

- $\alpha = 2/T$ (standard bilinear) corresponds to $\omega_0 = 0$.
- $\omega_0 = \pi/T$ is impossible.

Possible choices for ω_0:

- The cross-over frequency (which will help preserve the phase margin).
- The frequency of a critical notch.
- The frequency of a critical oscillatory mode.

The best choice depends on the most important features in your control design.

Remember: $C(s)$ stable implies $C(z)$ stable, but you **must** check that $\frac{1}{1 + P(z)C(z)}$ is stable!