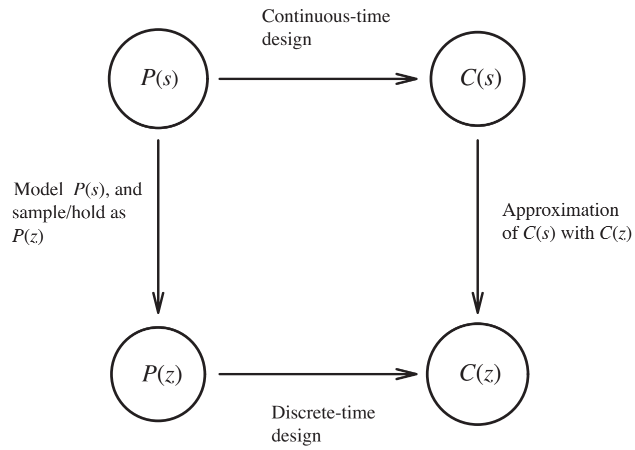
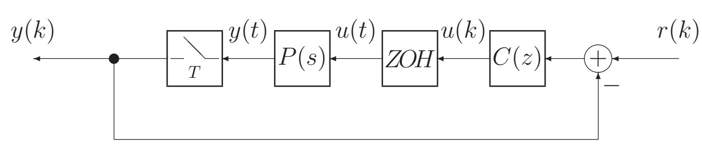
Modeling $P(z)$



- Develop a model of the discrete-time behavior of the plant.
- Allows digital designs to be performed directly.
- Evaluating the stability of the discrete-time system ($C(z)$ and $P(z)$ in feedback).

Sample and Hold Systems

The continuous-time plant, $P(s)$, is preceeding by a zero-order hold and followed by a sampler.



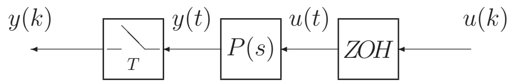
Performance and stability is specified in terms of the digital domain signals, $r(k)$, $y(k)$, $u(k)$, etc.

- Analog/Digital (A/D) board: sampler.
- Digital/Analog (D/A) board: zero-order hold.

Other options are possible but the above are by far the most common.

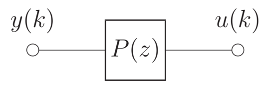
Sample and Hold Systems

Model the system from the *ZOH* block to the sampler:



$P(s)$ is an LTI system $\Longrightarrow$ the system from $u(k)$ to $y(k)$ is LSI.

It has an equivalent Z-transform, $P(z)$.



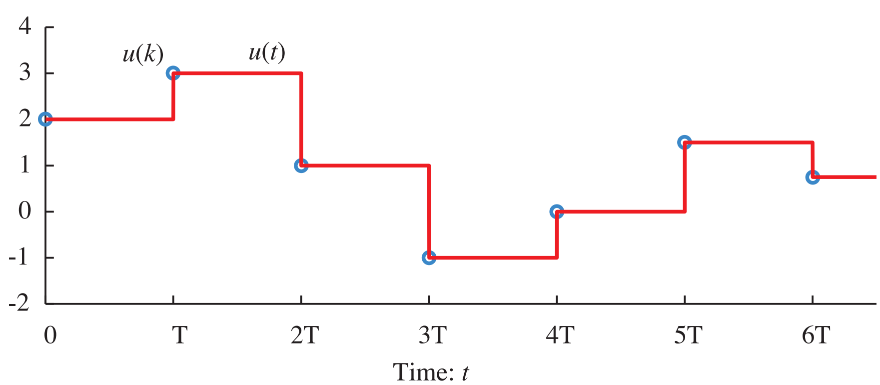
Zero-order hold equivalence: The closed-loop combination of $P(z)$ and $C(z)$ **exactly** models $P(s)$ in closed-loop at the sample times.

Zero-order hold equivalence

This is a reasonable model of a typical digital to analog (D/A) converter.

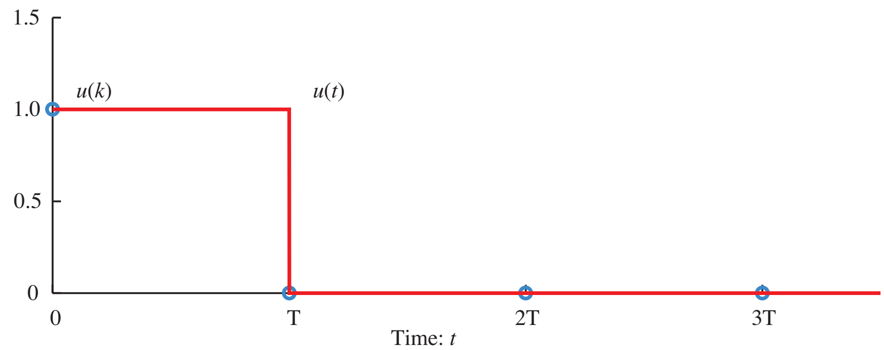
At the sample-time, $t = kT$, the discrete input, $u(k)$, is put on the output, $u(t)$. This value is held constant for the entire sample period. So,

$$u(t) = u(k), \quad \text{for} \quad kT \leq t < kT + T.$$



Pulse response

Pulse input: $u(k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$, gives the output $u(t) = \begin{cases} 0 & t<0 \\ 1 & 0 \leq t < T \\ 0 & t \geq T \end{cases}$.



Equivalently, the pulse response is:

$u(t) = \text{step}(t) - \text{step}(t-T)$,

(step denotes the unit step function).

Discrete-time transfer function

The discrete-time transfer function is the z-transform of the sampled pulse response.

For a pulse, $u(k)$, the plant input is,

$u(t) = \text{step}(t) - \text{step}(t-T)$.

The plant output (in the Laplace domain) is

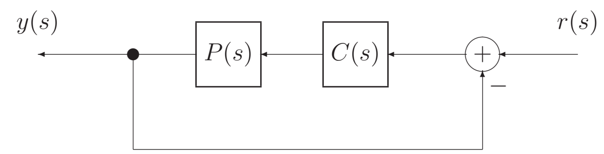
$$y(s) = \left(1 - e^{-Ts}\right) \frac{P(s)}{s}.$$

We now sample this, and take the Z-transform.

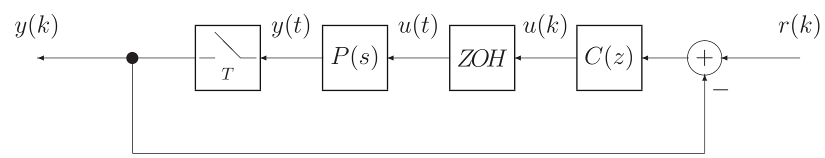
$$\begin{aligned} P(z) &= \mathcal{Z}\left\{ \left(1 - e^{-Ts}\right) \frac{P(s)}{s} \right\} = \mathcal{Z}\left\{ \frac{P(s)}{s} \right\} - \mathcal{Z}\left\{ e^{-Ts} \frac{P(s)}{s} \right\} = \mathcal{Z}\left\{ \frac{P(s)}{s} \right\} - z^{-1}\mathcal{Z}\left\{ \frac{P(s)}{s} \right\} \\ &= (1 - z^{-1})\mathcal{Z}\left\{ \frac{P(s)}{s} \right\}. \end{aligned}$$

Easily calculated via several MATLAB functions (`c2d` or `zohequiv`).

Design closed-loop (continuous-time)

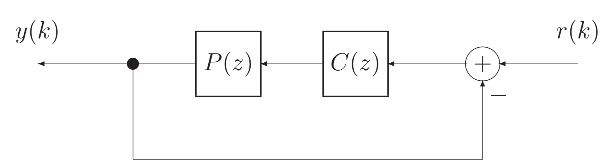


Implemented closed-loop. Note $C(z)$ approximates $C(s)$.



Stability/Performance evaluation

$P(z)$ is the zero-order hold equivalent of $P(s)$.



Remaining issues

- This analysis considers the system response at the sample times.
  - Hidden oscillations
  - Intersample behavior
- How should one modify $C(z)$ if the discrete-time response is not satisfactory?

An example

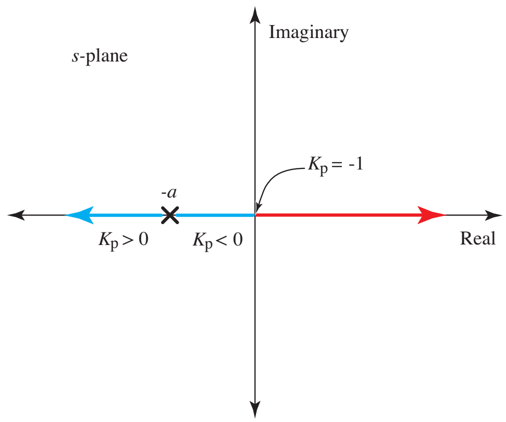
Consider a proportional controller: $C(s) = C(z) = K_p$.

And a simple plant: $P(s) = \dfrac{a}{s+a}, \qquad a > 0.$

Root locus

$P(s)K_p$ has one pole and no zeros.

The closed-loop, with $C(s) = K_p$, is theoretically stable for all $-1 < K_p \leq \infty$.



The example continued

Increasing K_p has the following effects:

- Decreasing the rise-time,
- Reducing the settling-time
- Reducing the steady-state tracking error.
- Increasing the controller output amplitude (more gain ⇒ more $).

In reality too much gain will eventually destabilize the continuous-time system (**why?**).

Digital implementation

ZOH equivalent for $P(s)$:

$$\begin{aligned} P(z) &= (1 - z^{-1})\mathcal{Z}\left\{ \frac{P(s)}{s} \right\} \\ &= (1 - z^{-1})\mathcal{Z}\left\{ \frac{a}{s(s+a)} \right\} \\ &= \frac{1 - e^{-aT}}{z - e^{-aT}}. \end{aligned}$$

$P(z)$ has a (stable) pole at $z = e^{-aT}$.

Approximation for $C(s)$:

$C(s) = K_p$ so $C(z) = K_p$

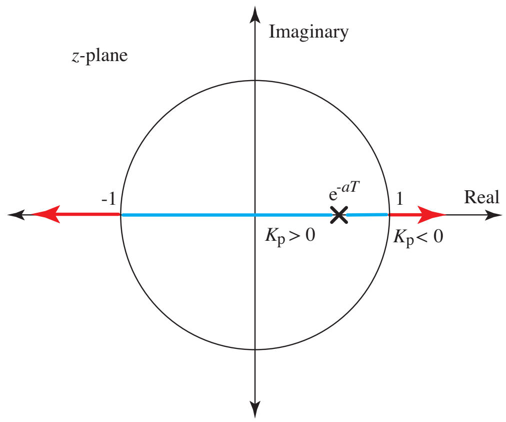
The example continued

Root locus (discete-domain)

$P(z)K_p$ has one pole and no zeros.

$1 + K_p P(z)$ is **not** stable
for all $K_p > 0$!

Characteristic equation:
$$1 + \frac{K_p(1 - e^{-aT})}{z - e^{-aT}} = 0,$$

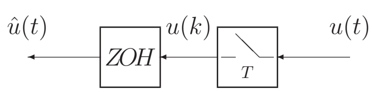


Closed-loop pole: $z = e^{-aT} - K_p(1 - e^{-aT})$. For $K_p > \dfrac{1 + e^{-aT}}{1 - e^{-aT}}$, the system is unstable.

The *ZOH* adds potentially destabilizing phase lag to the feedback loop

Effect of a sample and hold

Consider the simplest possible system: a sampler and a *ZOH*.

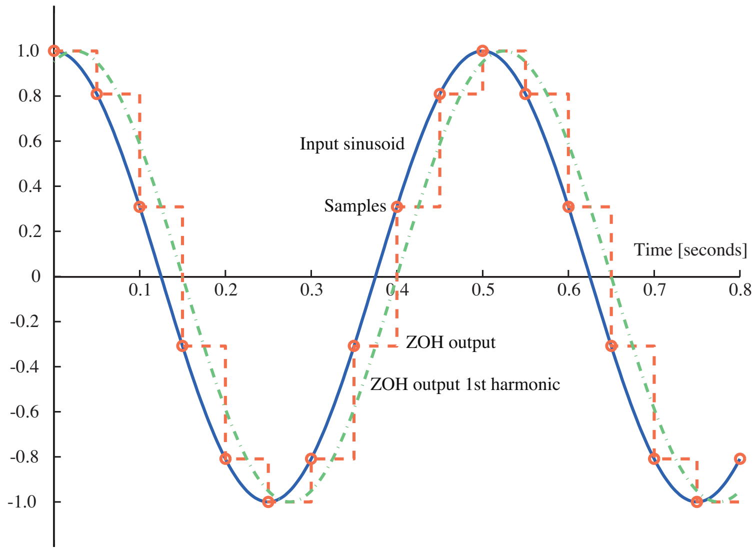


ZOH **Output:**

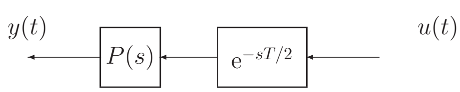
- contains high frequency components (input was a single frequency).
- has the fundamental frequency component shifted by $T/2$ seconds.

Is this system LTI?

Effect of a sample and hold



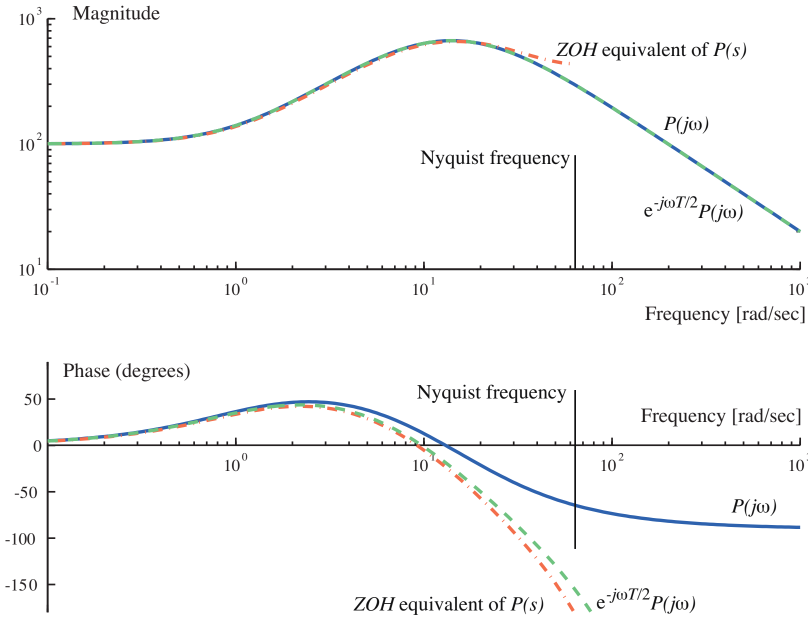
Precompensate by including a delay in the continuous design



The additional delay approximates the phase lag that the *ZOH* will introduce in the digital implementation.

If $C(s)$ is designed to work with $P(s)e^{-sT/2}$, then it will probably work reasonably well for a *ZOH* implementation of $P(s)$.

Precompensate by including a delay in the continuous design



Rational approximations to $\mathrm{e}^{-sT/2}$:

Padé Approximations $P(s)$:

First order lag: $\dfrac{1}{1 + sT/2}$

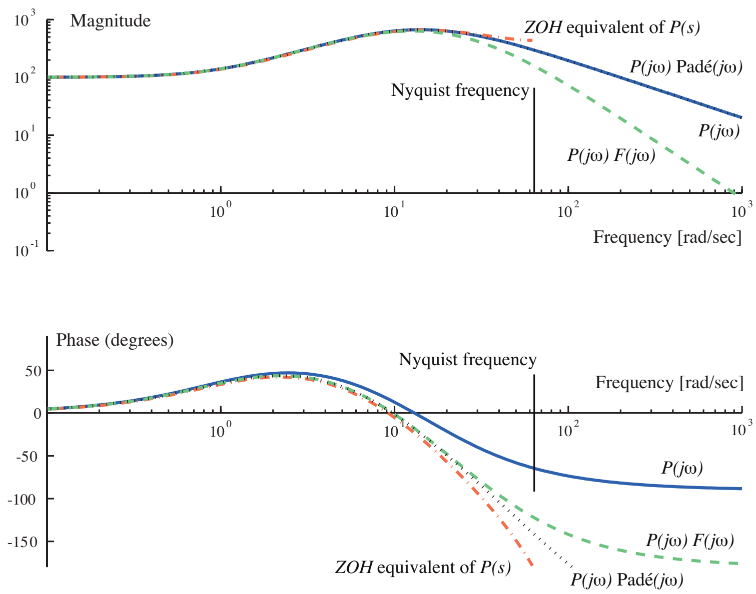
First order Padé: $\dfrac{1 - sT/4}{1 + sT/4}$

Nth order Padé: $\mathrm{e}^{-\theta s} \approx \dfrac{\left(1 - \frac{\theta}{2n}s\right)^n}{\left(1 + \frac{\theta}{2n}s\right)^n}$

We typically use a first order Padé approximation which adds one pole and one zero to the plant for our design of $C(s)$.

If the plant dynamics are close to the Nyquist frequency we may choose to use a second order Padé approximation for greater accuracy.

Rational approximations to $e^{-sT/2}$: $\quad F(s) = \dfrac{1}{1 + sT/2} \quad \text{Padé}(s) = \dfrac{1 - sT/4}{1 + sT/4}$



Exercise:

Predict the range of stability for $C(s) = K_p$ in the previous example where:

1. $P(s)$ is augmented with a first order lag.
2. $P(s)$ is augmented with a first order Padé approximation.

How do these stability ranges compare to the actual stability range for $P(z)K_p$?